Lump dynamics in the $\mathbb{C}P^1$ model on the torus

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Abstract

The topology and geometry of the moduli space, M_2, of degree 2 static solutions of the $\mathbb{C}P^1$ model on a torus (spacetime $T^2 \times \mathbb{R}$) are studied. It is proved that M_2 is homeomorphic to the left coset space G/G_0 where G is a certain eight-dimensional noncompact Lie group and G_0 is a discrete subgroup of order 4. Low energy two-lump dynamics is approximated by geodesic motion on M_2 with respect to a metric g defined by the restriction to M_2 of the kinetic energy functional of the model. This lump dynamics decouples into a trivial "centre of mass" motion and nontrivial relative motion on a reduced moduli space. It is proved that (M_2, g) is geodesically incomplete and has only finite diameter. A low dimensional geodesic submanifold is identified and a full description of its geodesics obtained.

1 Introduction

The $\mathbb{C}P^1$ model in $(2+1)$ dimensions has long been popular in theoretical physics, both for its condensed matter applications, and as a simple nonlinear field theory possessing topological solitons, usually called lumps. The Euler-Lagrange equation of the system is not integrable, so there is no hope of solving the multilump initial value problem exactly. Numerical simulations of the model have revealed a rich diversity in the lump dynamics, which includes not only the now-familiar 90° scattering in head on collisions, but also lump expansion, collapse and singularity formation. It is an interesting and highly nontrivial problem to understand the mechanisms underlying this complicated dynamics.

Such understanding has been afforded in similar field theories (those of Bogomol'nyi type) by the geodesic approximation of Manton [1, 2, 3]. Here the low-energy dynamics of n solitons is approximated by geodesic motion in the moduli space of *static* n-soliton solutions, M_n, the metric g being defined by the restriction to M_n of the kinetic energy functional of the field theory. So understanding n-soliton dynamics is reduced to studying the topology and geometry of (M_n, g), a finite dimensional, smooth Riemannian manifold.

Several authors have pursued this programme for the $\mathbb{C}P^1$ model in $\mathbb{R}^{2+1}$ with standard boundary conditions [4, 5], concentrating on the case of two lumps. There is, however, a technical problem: the metric on M_2 does not, strictly speaking, exist, that is, at every point $p \in M_2$ some vectors in T_pM_2 are assigned infinite length by the kinetic energy functional (they are "non-normalizable zero modes"). These divergences stem from the noncompactness of space $\mathbb{R}^2$. They are essentially due to the existence in the general static solution of scale and orientation parameters which are frozen in the geodesic approximation because to alter them, no matter how slowly, costs infinite kinetic energy. This is only possible because the kinetic energy is an integral over a noncompact space. One can study geodesic motion orthogonal to the bad directions, or one can remove the problem entirely by studying the model on a compact space [6]. In this paper we impose square periodic spatial boundary conditions on the model, or, equivalently, place it on a flat torus. The aim is to

establish rigorously results concerning the topology and geometry of (M_2, g), and to describe their implications for low-energy two lump dynamics on the torus, within the framework of the geodesic approximation. The work is arranged as follows.

In section 2 we introduce the $\mathbb{C}P^1$ model on the torus, and review some relevant background material. In particular we use a standard argument of Belavin and Polyakov to show that M_2 is the space of degree 2 elliptic functions.

In section 3 we equip M_2 with a natural metric topology, and prove that it is homeomorphic to the left coset space G/G_0, where G is the Lie group $PSL(2,\mathbb{C}) \times T^2$ and G_0 is discrete subgroup of order 4. This allows one to give M_2 a natural differentiable structure (that of the smooth manifold G/G_0) and provides M_2 with a good global parametrization, using the covering space G.

In section 4 this parametrization is used to survey the degree 2 static solutions and describe their energy density distributions. It is found that exceptionally symmetric solutions exist with four, rather than two identical energy lumps, as well as the expected two-lump and annular solutions.

In section 5 the metric g on M_2 is defined, and some of its properties discussed. We lift g to obtain $\widetilde{g}$, the metric on the covering space G, and show that $\widetilde{g}$ is a product metric on $PSL(2,\mathbb{C}) \times T^2$. In this way, we show that lump dynamics in the geodesic approximation decomposes into a trivial "centre of mass" motion, the T^2 part, and a nontrivial relative motion, the $PSL(2,\mathbb{C})$ part. So attention may be restricted to geodesic motion on a reduced covering space, without loss of generality.

In section 6 it is proved that (M_2, g) is geodesically incomplete by finding an explicit, maximally extended geodesic, and showing that it has only finite length. It follows that lumps can collapse to form singularities in finite time.

In section 7 a 2-dimensional totally geodesic submanifold is identified by computing the fixed point set of a discrete group of isometries. The geodesics of this submanifold and their associated lump motions are described.

In section 8 it is proved that (M_2, g) has only finite diameter, despite its noncompactness. One should therefore visualize it as having only finite extent. In consequence, all static solutions are close to the end of moduli space, that is, close to collapse.

In section 9 some concluding remarks are presented. Two 3-dimensional totally geodesic submanifolds are identified, and it is shown that 90° head on scattering must occur in the model under certain conditions. The present work is summarized, and extensions suggested.

2 The $\mathbb{C}P^1$ model on the torus

The field, a map from spacetime to $\mathbb{C}P^1$, $W : \mathbb{R} \times T^2 \to \mathbb{C}P^1$, will throughout be considered complex valued, so that we are using an inhomogeneous coordinate on $\mathbb{C}P^1$, or equivalently, a stereographic coordinate on S^2, exploiting the well known diffeomorphism between $\mathbb{C}P^1$ and the two sphere. The metric and volume form on the codomain in terms of such a coordinate are, respectively,

$$h = \frac{4\,du\,d\bar{u}}{(1+|u|^2)^2} \qquad \omega = \frac{2i\,du \wedge d\bar{u}}{(1+|u|^2)^2}. \tag{1}$$

It is convenient to use a complex coordinate on physical space also, by identifying T^2 with $\mathbb{C}/\Omega$ where Ω is the period module, which we choose, for concreteness, to be

$$\Omega = \{n + im : n, m \in \mathbb{Z}\}. \tag{2}$$

So we impose square periodic boundary conditions of unit period on W. Position in T^2 is parametrized by position $z = x + iy$ in the covering space $\mathbb{C}$. The metric on spacetime is $\eta = dt^2 - dx^2 - dy^2$, and the action functional of the field theory is the standard harmonic map functional for mappings $(\mathbb{R} \times T^2, \eta) \to (\mathbb{C}P^1, h)$, that is,

$$S[W] = \int_{\mathbb{R}\times T^2} \frac{\partial_\mu W \partial_\nu \bar{W}}{(1+|W|^2)^2} \eta^{\mu\nu}. \tag{3}$$

This may be written in a fashion reminiscent of Lagrangian mechanics, $S = \int dt(T - V)$, upon definition of the kinetic and potential energy functionals,

$$
\begin{aligned}
T &= \int_{T^2} \frac{|\dot{W}|^2}{(1+|W|^2)^2} & (4) \\
V &= \int_{T^2} \frac{1}{(1+|W|^2)^2} \left(\left| \frac{\partial W}{\partial x} \right|^2 + \left| \frac{\partial W}{\partial y} \right|^2 \right). & (5)
\end{aligned}
$$

The configuration space Q is $C^1(T^2, S^2)$, the space of continuously differentiable maps $T^2 \to S^2$ (note that $V[W]$ is finite for all $W \in Q$ by compactness of T^2). By Hopf's Degree Theorem [7], Q decomposes into disjoint homotopy classes labelled by topological degree n, an integer,

$$
Q = \coprod_{n \in \mathbb{Z}} Q_n. \tag{6}
$$

Physically, n is interpreted as the "lump number" of the configuration, the excess of lumps over antilumps.

Static solutions are extremals of V, that is harmonic maps $T^2 \to S^2$. The space of minimal energy static solutions in Q_n is called the degree n moduli space, denoted M_n. A well-known argument due to Belavin and Polyakov [8] shows that M_n (n assumed nonnegative) is in fact the space of degree n elliptic functions, that is, holomorphic maps $T^2 \to S^2$:

$$
\begin{aligned}
0 &\leq \int_{T^2} \frac{|\partial_x W + i\partial_y W|^2}{(1+|W|^2)^2} \\
&= V[W] - \frac{1}{2} \int_{T^2} W^* \omega \\
&= V[W] - \frac{1}{2} \mathrm{Vol}(S^2) n = V[W] - 2\pi n & (7)
\end{aligned}
$$

where $W^*\omega$ is the pullback of the volume from on S^2 by W. It follows that

$$
V|_{Q_n} \geq 2\pi n \tag{8}
$$

with equality if and only if $(\partial_x + i\partial_y)W = 0$, which is the Cauchy-Riemann equation for W. So if there exist degree n elliptic functions, then M_n is the space of such functions, since any other function has higher energy. If there are no such functions, then M_n is empty, for the energy bound (8) is optimal. To see this, consider the following family of functions. For $\epsilon > 0$ small, define $W_\epsilon \in Q_n$ so that

$$
W_\epsilon(z) = \begin{cases} \frac{\epsilon^{2n}}{z^n} & |z| < \epsilon \\ \\ 0 & |z| > 2\epsilon \end{cases} \tag{9}
$$

interpolating between these two regions with a smooth cutoff function. This consists of a flat-space degree n lump of width ϵ^2 cut off on a disc of radius ϵ. Since W_ϵ is not exactly holomorphic, $V[W_\epsilon] > 2\pi n$, but the excess can be made arbitrarily small by choosing ϵ small enough.

It is easily proved that there are no unit degree elliptic functions [9], so we conclude that $M_1 = \emptyset$, and the simplest nontrivial moduli space is M_2.

3 The degree two moduli space

Weierstrass explicitly constructed a degree 2 elliptic function $\wp$, and it is on this that we base our parametrization of M_2. The partial fraction representation of $\wp$ is

$$
\wp(z) = \frac{1}{z^2} - \sum_{\nu \in \Omega \setminus \{0\}} \left[\frac{1}{(z-\nu)^2} - \frac{1}{\nu^2} \right]. \tag{10}
$$

Several properties of $\wp$ will be needed, some of which follow easily from equation (10), others of which are less straightforward. A comprehensive treatment can be found in [10]. Specifically:

$$\wp(iz) = -\wp(z), \qquad \wp(-z) = \wp(z), \qquad \wp(\bar{z}) = \overline{\wp(z)},$$
$$\wp'(z)^2 = 4\wp(z)(\wp(z)^2 - e_1^2), \tag{11}$$

where $e_1 = \wp(\frac{1}{2})$ is a real number, approximately 6.875. It follows that $\wp$ is real on the boundary and central cross of the unit square, and purely imaginary on the diagonals of the unit square (see figure 1) and that $\wp$ has a double pole at 0 and a double zero at $(1+i)/2$.

Given one holomorphic function $\wp \in M_2$ one can obtain others by composing on the right with a rigid translation of T^2 and on the left with a Möbius transformation of S^2 since these preserve holomorphicity and degree. In terms of a stereographic coordinate W on S^2, Möbius transformations are unit degree rational maps [11]

$$W \mapsto \frac{a_{11}W + a_{12}}{a_{21}W + a_{22}} \tag{12}$$

where $a_{ij} \in \mathbb{C}$ and $a_{11}a_{22} \neq a_{12}a_{21}$ else the degree degenerates to zero. One may collect the parameters a_{ij} into a matrix $M \in GL(2,\mathbb{C})$ and denote the action of the matrix M on S^2 defined in equation (12) by $W \mapsto M \odot W$. (The constraint $a_{11}a_{22} \neq a_{12}a_{21}$ is now $\det M \neq 0$, ensuring that M is invertible, and hence in $GL(2,\mathbb{C})$.) Composition of Möbius transformations coincides with matrix multiplication,

$$M_2 \odot (M_1 \odot W) \equiv (M_2 M_1) \odot W. \tag{13}$$

Note, however, that this Möbius representation of $GL(2,\mathbb{C})$ is not faithful since any pair of matrices $M, M' \in GL(2,\mathbb{C})$ such that $M = \lambda M'$ for some $\lambda \in \mathbb{C}$ generate the same Möbius transformation. Denoting this scale equivalence $\sim$ we identify the Möbius group with $GL(2,\mathbb{C})/\sim$, each equivalence class of which may be represented by a unimodular matrix ($\det M = 1$). If M is unimodular then so is $-M$, so $SL(2,\mathbb{C})$ is a double cover of $GL(2,\mathbb{C})/\sim$, and the Möbius group is identified with $SL(2,\mathbb{C})/\mathbb{Z}_2$, usually denoted $PSL(2,\mathbb{C})$, which is easily seen to be six dimensional (the P stands for "projective").

For the sake of brevity, let G denote the eight dimensional Lie group $PSL(2,\mathbb{C}) \times T^2$ with the group product $(M_1, s_1)\cdot(M_2, s_2) = (M_1 M_2, s_1 + s_2)$. We can define a G-action on M_2, $G \times M_2 \to M_2$ such that $(g, W) \mapsto W_g$ where

$$W_{(M,s)}(z) = M \odot W(z - s). \tag{14}$$

We claim that this action is transitive, since the G-orbit of $\wp$ exhausts M_2.

Lemma 1 *For each $W \in M_2$ there exists $(M, s) \in G$ such that $W(z) = M \odot \wp(z - s)$.*

Proof: This may be established in several ways [12, 13, 14]. One economical, instructive (and apparently novel) argument appeals to the Riemann-Hurwitz formula, which constrains the number and valency of multivalent points of a holomorphic mapping between compact Riemann surfaces given their genera and the degree of the map [15]. In the case of a degree 2 holomorphic map from T^2 to S^2 the formula states that any such function must have exactly 4 distinct double valency points (for $\wp$ these are $0, \frac{1}{2}, \frac{i}{2}$ and $(1+i)/2$).

Let $W \in M_2$ and $s \in T^2$ be one of its double valency points which is not a double pole. Then $(W(z+s) - W(s))^{-1}$ is another elliptic function with a double pole at 0, and no poles elsewhere (in the fundamental period square). Its Laurent expansion about 0 is

$$\frac{1}{W(z+s) - W(s)} = \frac{a_1}{z^2} + \frac{a_2}{z} + a_3 + \cdots \tag{15}$$

where $a_1 \neq 0$. Consider $f(z) = [W(z+s) - W(s)]^{-1} - a_1\wp(z)$. This is an elliptic function with at most a simple pole at 0, and no poles elsewhere. Hence it has degree 1 or degree 0. But there are no degree 1 elliptic functions, and all degree 0 elliptic functions are constant, so $f(z) = c$. Defining

$$M' = \begin{pmatrix} a_1 W(s) & cW(s)+1 \\ a_1 & c \end{pmatrix} \tag{16}$$

and $M = (\det M')^{-\frac{1}{2}} M'$, it follows that $W(z) = M \odot \wp(z-s)$. $\Box$

It is clear that for each $W \in M_2$ the associated $g \in G$ is not unique, since any one of the four distinct double valency points can be chosen as the basis of the construction of (M, s) outlined above. Conversely, given a choice of $s \in T^2$, a double valency point of W, the construction of $M \in PSL(2,\mathbb{C})$ is unique, so for each $W \in M_2$ there are exactly four different $g \in G$ such that $W = \wp_g$. In particular, we can construct three alternative formulae for $\wp(z)$ based on the three double valency points $s_0 = (1+i)/2$, $s_1 = 1/2$ and $s_2 = i/2$ (the trivial formula $\wp(z)$ results from choosing $s = 0$, the fourth double valency point):

$$\wp(z) \equiv \frac{-e_1^2}{\wp(z-s_0)} \tag{17}$$

$$\equiv \frac{e_1[\wp(z-s_1)+e_1]}{\wp(z-s_1)-e_1} \tag{18}$$

$$\equiv \frac{-e_1[\wp(z-s_2)-e_1]}{\wp(z-s_2)+e_1}. \tag{19}$$

These are found by computing the Laurent expansions of $\wp$ about s_i using (11), the formula for $\wp'$. It is convenient to treat M_2 as the G-orbit of $\wp/e_1$, rather than $\wp$. The identities (17,18,19) can be rewritten

$$\frac{\wp(z)}{e_1} \equiv U_i \odot \left(\frac{\wp(z-s_i)}{e_1}\right) \qquad i = 0, 1, 2 \tag{20}$$

where U_i are the following $SU(2)$ matrices:

$$U_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad U_1 = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad U_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{21}$$

So the stabilizer of $\wp/e_1$ under the G-action is

$$G_0 = \{(I,0), (U_0, s_0), (U_1, s_1), (U_2, s_2)\}, \tag{22}$$

a discrete subgroup of G isomorphic to the Viergruppe V_4, that is, abelian, with each element its own inverse (when checking this recall that $SL(2,\mathbb{C})$ matrices which differ only in sign are identified). The $SU(2)$ subgroup of $SL(2,\mathbb{C})$ acting on S^2 via $\odot$ is a double cover of $SO(3)$ acting on S^2 via the natural rotation action. So G_0 is a discrete group of simultaneous rotations of the target space S^2 and translations of the domain T^2. In fact a straightforward calculation shows that the U_i are rotations of S^2 by π about three orthogonal axes.

For a general $W \in M_2$, then, if $W = (\wp/e_1)_g$ then $W = (\wp/e_1)_h$ if and only if h is an element of the left coset gG_0, which we will henceforth denote $[g]$. So the mapping $\phi : G/G_0 \to M_2$, $\phi : [g] \mapsto \phi_{[g]}$ where

$$\phi_{[(M,s)]}(z) = M \odot \left(\frac{\wp(z-s)}{e_1}\right) \tag{23}$$

is well defined and bijective. It would seem natural, therefore, to identify M_2 with G/G_0 via ϕ, but this only makes sense provided ϕ is a homeomorphism. Before proving that this is indeed the case, there are a few necessary preliminaries. Let $p : G \to G/G_0$ be the projection map $p(g) = [g]$. Since

G_0 is a discrete subgroup of G, it acts freely and properly discontinuously on G, so the quotient space G/G_0 is, like G itself, a Hausdorff, smooth manifold [16]. The pair (G, p) is a covering space of G/G_0, and p is a local homeomorphism.

It is useful to define $\widetilde{\phi} : G \to M_2$ such that $\widetilde{\phi} = \phi \circ p$, that is, $\widetilde{\phi} : g \mapsto \widetilde{\phi}_g = (\wp/e_1)_g$. The Lie group $SL(2, \mathbb{C})$ is noncompact, and is, in fact, homeomorphic to $\mathbb{R}^3 \times SU(2)$, as may be shown [17] by decomposing any $SL(2, \mathbb{C})$ matrix M into the product HU, where $U \in SU(2)$ and H is a positive definite, hermitian, unimodular matrix, this pair being unique. The space of H-matrices is homeomorphic to $\mathbb{R}^3$ and may be parametrized so that for all $\boldsymbol{\lambda} \in \mathbb{R}^3$,

$$H(\boldsymbol{\lambda}) = \sqrt{1 + |\boldsymbol{\lambda}|^2} I + \boldsymbol{\lambda} \cdot \boldsymbol{\tau}, \tag{24}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli spin matrices. It follows that $PSL(2, \mathbb{C}) \cong \mathbb{R}^3 \times (SU(2)/\mathbb{Z}_2) \cong \mathbb{R}^3 \times SO(3)$, and so $G \cong \mathbb{R}^3 \times SO(3) \times T^2$.

To prove that ϕ is a homeomorphism we will need to understand the behaviour of $\widetilde{\phi}_g : T^2 \to S^2$ as g approaches the end of G, i.e. as $\lambda = |\boldsymbol{\lambda}| \to \infty$. For this purpose, consider the one parameter family $\{\phi_{\lambda, \widehat{\lambda}} = \widetilde{\phi}_{(M,0)} \in M_2 : M = H(\lambda \widehat{\boldsymbol{\lambda}}), \lambda \in (0, \infty)\}$ for some fixed $\widehat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}/\lambda \in S^2$. Explicitly,

$$\phi_{\lambda, \widehat{\lambda}}(z) = H(\boldsymbol{\lambda}) \odot \left(\frac{\wp(z)}{e_1} \right). \tag{25}$$

The action of $H(\boldsymbol{\lambda})$ on S^2 ($H(\boldsymbol{\lambda}) : W \mapsto H(\boldsymbol{\lambda}) \odot W$) has exactly two fixed points, $\widehat{\boldsymbol{\lambda}}$ and $-\widehat{\boldsymbol{\lambda}}$, and as $\lambda \to \infty$ all but a vanishing neighbourhood of $-\widehat{\boldsymbol{\lambda}}$ is mapped by $H(\boldsymbol{\lambda})$ to within a vanishing neighbourhood of $\widehat{\boldsymbol{\lambda}}$ [6]. So the limiting function $\phi_{\infty, \widehat{\lambda}} : T^2 \to S^2$ has the general form

$$\phi_{\infty, \widehat{\lambda}}(z) = \lim_{\lambda \to \infty} \phi_{\lambda, \widehat{\lambda}}(z) = \begin{cases} \widehat{\boldsymbol{\lambda}} & z \notin (\wp/e_1)^{-1}(-\widehat{\boldsymbol{\lambda}}) \\ -\widehat{\boldsymbol{\lambda}} & z \in (\wp/e_1)^{-1}(-\widehat{\boldsymbol{\lambda}}). \end{cases} \tag{26}$$

That is, for generic $\widehat{\boldsymbol{\lambda}}$, all but two points of T^2, the preimages of $-\widehat{\boldsymbol{\lambda}}$ under $\wp/e_1$, are mapped by $\phi_{\infty, \widehat{\lambda}}$ to $\widehat{\boldsymbol{\lambda}}$, while these two points are mapped to $-\widehat{\boldsymbol{\lambda}}$. In the four special cases $\widehat{\boldsymbol{\lambda}} \in \{(0, 0, \pm 1), (\pm 1, 0, 0)\}$, the preimages of $-\widehat{\boldsymbol{\lambda}}$ coincide (double valency points) so all but one point in T^2 is mapped to $\widehat{\boldsymbol{\lambda}}$. The point to note is that in all cases $\phi_{\lambda, \widehat{\lambda}}$ collapses to a discontinuous limit.

The statement that $\phi : G/G_0 \to M_2$ is a homeomorphism is, of course, meaningless until we equip M_2 with a topology (the domain inherits its topology from G, which we take to have the natural product topology on $PSL(2, \mathbb{C}) \times T^2$). There are many sensible choices for the topology on T^2. One simple and directly physical choice is to endow Q_2 with the metric topology where distance between configurations is measured by their maximum pointwise deviation in the codomain S^2, so that $M_2 \subset Q_2$ inherits the relative topology. That is, let $d : S^2 \times S^2 \to \mathbb{R}$ be the usual distance function on S^2, and define $D : Q_2 \times Q_2 \to \mathbb{R}$ such that, for all $W_1, W_2 \in Q_2$,

$$D(W_1, W_2) = \sup_{z \in T^2} d(W_1(z), W_2(z)). \tag{27}$$

It is straightforward to verify that D satisfies the axioms of a distance function. The resulting metric topology on Q_2 is Hausdorff, as is any metric topology [18]. Rather than break up the smooth manifold G into coordinate charts, it is convenient to equip G with a metric topology also, as follows: let h be the (Riemannian) product metric

$$h = (d\boldsymbol{\lambda} \cdot d\boldsymbol{\lambda}) \oplus h_{SO(3)} \oplus ds\, d\bar{s} \tag{28}$$

on $G \cong \mathbb{R}^3 \times SO(3) \times T^2$, where $h_{SO(3)}$ is the biinvariant metric on $SO(3)$ of unit volume. The Riemannian manifold (G, h) has a natural distance function $\widetilde{d}$ where $\widetilde{d}(g_1, g_2)$ is the infimum of

lengths (with respect to h) of piecewise C^1 paths connecting g_1 and g_2. That $\widetilde{d}$ is a distance function, and that the associated metric topology coincides with the original topology on G (independent of the choice of h) are standard theorems of Riemannian geometry [19]. We may now state and prove Theorem 1. Throughout, $B_\epsilon(x)$ denotes the open ball of radius ϵ centred on x, where the space containing x (S^2, M_2 or G), and hence the appropriate distance function (d, D or $\widetilde{d}$), should be clear from context.

Theorem 1 *The bijection $\phi : G/G_0 \to M_2$ is a homeomorphism.*

Proof: We must prove that both ϕ and ϕ^{-1} are continuous. To prove the former, it suffices to show that $\widetilde{\phi} = \phi \circ p$, is continuous, since the projection p is a local homeomorphism. Fix $g_0 \in G$ and $\epsilon > 0$. Then we must show that $\exists \delta > 0$ such that $\forall g \in B_\delta(g_0)$, $\widetilde{\phi}_g \in B_\epsilon(\widetilde{\phi}_{g_0})$. Let $\phi_* : G \times T^2 \to S^2$ such that $\phi_*(g, z) = \widetilde{\phi}_g(z)$. Note that ϕ_* is manifestly continuous. Hence, for each $\tilde{z} \in T^2$ there exists $\delta(\tilde{z}) > 0$ such that

$$(g, z) \in B_{\delta(\tilde{z})}(g_0) \times B_{\delta(\tilde{z})}(\tilde{z}) \Rightarrow d(\phi_*(g, z), \phi_*(g_0, \tilde{z})) < \frac{\epsilon}{3}. \tag{29}$$

The collection of open balls $\{B_{\delta(z)} \subset T^2 : z \in T^2\}$ is an open cover of T^2. Since T^2 is compact, there exists a finite subcover $\{B_{\delta(z_j)}(z_j) : j = 1, 2, \ldots, N\}$. Define $\delta = \inf\{\delta(z_j) : j = 1, 2, \ldots, N\} > 0$.

Now, let $g \in B_\delta(g_0)$ and consider $D(\widetilde{\phi}_g, \widetilde{\phi}_{g_0})$. For each $z \in T^2$ there exists $j \in \{1, 2, \ldots, N\}$ such that $z \in B_{\delta(z_j)}(z_j)$. Further, $g, g_0 \in B_\delta(g_0) \subset B_{\delta(z_j)}(g_0)$ by definition of δ, so $(g, z), (g_0, z) \in B_{\delta(z_j)}(g_0) \times B_{\delta(z_j)}(z_j)$. Hence, using (29) and the triangle inequality,

$$d(\phi_*(g, z), \phi_*(g_0, z)) \leq d(\phi_*(g, z), \phi_*(g_0, z_j)) + d(\phi_*(g_0, z), \phi_*(g_0, z_j)) < \frac{2\epsilon}{3}. \tag{30}$$

Thus,

$$D(\widetilde{\phi}_g, \widetilde{\phi}_{g_0}) = \sup_{z \in T^2} d(\phi_*(g, z), \phi_*(g_0, z)) \leq \frac{2\epsilon}{3} < \epsilon, \tag{31}$$

so $\widetilde{\phi}$ is continuous.

To prove that ϕ^{-1} is continuous we again convert the problem to one involving $\widetilde{\phi}$, using general properties of covering spaces. Fix $[g_0] \in G/G_0$ and choose any open neighbourhood U of $[g_0]$. The inverse image of $[g_0]$ under p is the left coset $g_0 G_0 = \{g_0, g_1, g_2, g_3\}$. Since p is a local homeomorphism there exists $\epsilon > 0$ such that $p(U_\epsilon) \subset U$, where

$$U_\epsilon := \bigcup_{i=0}^{3} B_\epsilon(g_i) \subset G. \tag{32}$$

We will show that there exists $\delta > 0$ such that $\widetilde{\phi}^{-1}(B_\delta(W_0)) \subset U_\epsilon$, where $W_0 = \phi_{[g_0]} \in M_2$. It follows that $\phi^{-1}(B_\delta(W_0)) \subset U$, and hence that ϕ^{-1} is continuous.

For each $n \in \mathbb{N}$, define the compact set

$$A_n = \overline{B}_n(I, 0) \backslash U_\epsilon \subset G \tag{33}$$

where $\overline{B}_n(I, 0)$ is the closed ball of radius n centred on $(I, 0) \in G$. Since $\widetilde{\phi}$ is continuous, $\widetilde{\phi}(A_n) \subset M_2$ is also compact, and therefore closed (M_2 is Hausdorff). Hence the complement $M_2 \backslash \widetilde{\phi}(A_n)$ is open, and it contains W_0 by construction (since $A_n \cap U_\epsilon = \emptyset$), so there exists $\delta_n > 0$ such that $B_{\delta_n}(W_0) \subset M_2 \backslash \widetilde{\phi}(A_n)$. In this way, construct a positive sequence $(\delta_n)_{n=1}^\infty$, which, without loss of generality, we may assume is decreasing and converges to 0. Consider the preimage of $B_{\delta_n}(W_0)$ under $\widetilde{\phi}$. By construction, $\widetilde{\phi}^{-1}(B_{\delta_n}(W_0)) \cap A_n = \emptyset$, so every point in the preimage lies either in U_ϵ, or at a distance greater than n from $(I, 0) \in G$.

We claim that there exists $N \in \mathbb{N}$ such that $\widetilde{\phi}^{-1}(B_{\delta_n}(W_0)) \subset U_\epsilon$. Choosing $\delta = \delta_N$, the proof is then complete. Assume this claim is false. Then $\forall n \in \mathbb{N}$ there exists $g_n \notin \overline{B}_n(I, 0)$ such that $\widetilde{\phi}_{g_n} \in B_{\delta_n}(W_0)$. For each n, choose such a g_n and consider the sequence $(g_n)_{n=1}^\infty$. Since $(\delta_n)_{n=1}^\infty \to 0$, the image of the sequence under $\widetilde{\phi}$, $(\widetilde{\phi}_{g_n})_{n=1}^\infty$ converges to W_0 in M_2. Define two projection maps on $G \cong \mathbb{R}^3 \times SO(3) \times T^2$:

$$\begin{aligned} \pi_1 : G \to [0, \infty) \qquad & \text{such that} \qquad \pi_1(\boldsymbol{\lambda}, U, s) = \lambda = |\boldsymbol{\lambda}| \\ \pi_2 : G \to S^2 \times SO(3) \times T^2 \qquad & \text{such that} \qquad \pi_2(\boldsymbol{\lambda}, U, s) = (\widehat{\boldsymbol{\lambda}}, U, s). \end{aligned} \tag{34}$$

The singularity of π_2 when $\lambda = 0$ is irrelevant here. By construction, $(\lambda_n)_{n=1}^\infty = (\pi_1(g_n))_{n=1}^\infty$ is unbounded, and without loss of generality, we may choose g_n such that λ_n is increasing. Since $(\pi_2(g_n))_{n=1}^\infty$ takes values in a compact space, it has, by the Bolzano-Weierstrass theorem, a convergent subsequence $(\pi_2(g_{n_r}))_{r=1}^\infty$. By translation and rotation symmetry of T^2 and S^2 respectively, we may assume without loss of generality that its limit is $(\widehat{\boldsymbol{\lambda}}, I, 0)$.

Consider the image under $\widetilde{\phi}$ of the associated subsequence $(g_{n_r})_{r=1}^\infty$, which approaches the end of $\mathbb{R}^3 \times SO(3) \times T^2$ asymptotic to the line $\{(t\widehat{\boldsymbol{\lambda}}, I, 0) : t \in (0, \infty)\}$. The function $\widetilde{\phi}_{g_{n_r}}(z)$ converges pointwise, as $r \to \infty$, to $\phi_{\infty, \widehat{\lambda}}(z)$, the limiting function previously described (to check this, use continuity of the $SO(3)$ and T^2 actions on S^2 and T^2 respectively, and of the function $\wp/e_1$). But $\phi_{\infty, \widehat{\lambda}}$, being discontinuous, cannot be in M_2, and hence cannot be W_0, a contradiction. $\Box$

4 Degree 2 static solutions

An immediate corollary of Theorem 1 is that $(G, \widetilde{\phi})$ is a covering space of M_2. The aim of this section is to describe the connexion between any point $g \in G$ and its corresponding static solution $\widetilde{\phi}_g \in M_2$, that is, to obtain a picture of what the static lumps look like, and how they change as g varies. A configuration W may be visualized as a distribution of unit length three-vectors ("arrows") over the torus. The energy density function of W is

$$\mathcal{E}(x, y) = \frac{|W_x|^2 + |W_y|^2}{(1 + |W|^2)^2}, \tag{35}$$

so the energy is located where the direction of the arrows is varying sharply in (x, y), in other words, where neighbouring arrows are stretched apart. It is the function $\mathcal{E}$ that we will describe as W varies in M_2.

For this purpose, rather than using the hermitian-unitary (or "polar") decomposition of $SL(2, \mathbb{C})$ used above, another standard decomposition is convenient. Namely, any $M \in SL(2, \mathbb{C})$ may be uniquely decomposed into a product UT with $U \in SU(2)$ and T upper triangular, real on the diagonal, positive definite and unimodular. The space of such T-matrices is homeomorphic to $\mathbb{R}^+ \times \mathbb{C}$ (here $\mathbb{R}^+ = (0, \infty)$) and may be parametrized thus:

$$T(\alpha, \rho) = \begin{pmatrix} \sqrt{\alpha e_1} & \sqrt{\alpha e_1}\rho \\ 0 & 1/\sqrt{\alpha e_1} \end{pmatrix}. \tag{36}$$

This allows one to write any $W \in M_2$ in the form

$$W(z) = (UT) \odot \left(\frac{\wp(z - s)}{e_1} \right) = U \odot [\alpha(\wp(z - s) + \rho)]. \tag{37}$$

Changing $U \in SU(2)$ merely produces a global internal rotation of the solution and so has no effect on $\mathcal{E}(z)$. Changing $s \in T^2$ translates the solution on the torus, so it suffices to examine the three parameter family

$$W(z) = \alpha(\wp(z) + \rho) \tag{38}$$

$(\alpha, \rho) \in \mathbb{R}^+ \times \mathbb{C}$, whose energy density is

$$\mathcal{E}(z) = \frac{8\alpha^2 |\wp(z)||\wp(z)^2 - e_1^2|}{(1 + \alpha^2 |\wp(z) + \rho|^2)^2}. \tag{39}$$

Note that for all (α, ρ), $\mathcal{E} = 0$ at the four double valency points $z = 0, s_0, s_1, s_2$, around which the direction of the arrows is constant to first order.

The behaviour of $\mathcal{E}$ as (α, ρ) covers $\mathbb{R}^+ \times \mathbb{C}$ is remarkably varied, going beyond the two-lump and annular structures one might expect by analogy with the planar $\mathbb{C}P^1$ model. First, consider the case $\alpha = 1$. Here, the energy is located in lumps close to the two roots of $\wp(z) + \rho$ (symmetrically placed about s_0 since $\wp$ is even) where the denominator of (39) is smallest. The only exceptions are when these roots coincide, $\rho = 0, -e_1, e_1$, or are close to coincidence, for then the lumps lose their individual identity and form a ring-like structure (centred on s_0, s_1 or s_2 respectively) rather reminiscent of coincident planar solitons (figure 2). If we now imagine increasing α above 1, the effect on W is to pull all the arrows in the configuration towards the north pole of S^2 ($W = \infty$), so that those close to the south pole ($W = 0$) are stretched apart. Since the energy is located where the arrows are stretched apart, increasing α therefore tends to concentrate $\mathcal{E}$ more strongly on roots of $\wp(z) + \rho$, and the lumps become taller and narrower. As $\alpha \to \infty$ the lumps collapse and "pinch off". Conversely, if α is decreased below 1 the arrows of the configuration are pulled southwards, and for α very small, $\mathcal{E}$ concentrates on the double pole of $\wp(z) + \rho$ ($z = 0$), where W points north. In this case a ring structure appears, centred on $z = 0$, and collapses to zero width as $\alpha \to 0$. These two cases are compared in figure 3. Noting the symmetry property $\wp(iz) \equiv -\wp(z)$, we see that whenever ρ passes through $0 \in \mathbb{C}$ along a smooth curve, the roots of $\wp(z) + \rho$ coalesce and emerge at right angles to their line of approach, giving a first hint that the familiar $90°$ scattering of lumps through a ring structure may take place in the geodesic approximation. We shall return to this point later.

The special case $\rho = 0$ is exceptional, and will be prominent in later sections. Examining the formula (39) in this case, we see that the global maxima of $\mathcal{E}$ must occur where $\wp$ is purely imaginary. If not, assume that a global maximum occurs at z_0 and let $\wp(z_0) = u \in \mathbb{C} \backslash i\mathbb{R}$. Then

$$\mathcal{E}(z_0) = \frac{8\alpha^2 |u||u^2 - e_1^2|}{(1 + \alpha^2 |u|^2)^2} < \frac{8\alpha^2 |u|(|u|^2 + e_1^2)}{(1 + \alpha^2 |u|^2)^2} \tag{40}$$

where the inequality is strict since u^2 is not real-negative. But there exists $z_1 \in T^2$ such that $\wp(z_1) = i|u|$, and

$$\mathcal{E}(z_1) = \frac{8\alpha^2 |u|| - |u|^2 - e_1^2|}{(1 + \alpha^2 |u|^2)^2} > \mathcal{E}(z_0), \tag{41}$$

a contradiction. Given the symmetry of $\mathcal{E}$ under $\wp \mapsto -\wp$, and that $\wp$ is even, it follows that $\mathcal{E}(z)$ has at least four peaks on the diagonals of the unit square, symmetrically placed about s_0. Plots of $\mathcal{E}$ confirm that there are, in fact, exactly four such peaks (figure 4). The must symmetric case is $\alpha = 1/e_1$, that is, $W(z) = \wp(z)/e_1$. Here, using the identity (17), one can easily show that $\mathcal{E}(z - s_0) \equiv \mathcal{E}(z)$, so the four peaks are located halfway towards the centre s_0 along the diagonals, i.e. at the points $(1+i)/4, (3+i)/4, 3(1+i)/4, (1+3i)/4$. This solution has the most evenly spread energy distribution possible. Once again, one can consider the effect of increasing α (pulling the arrows northwards) or decreasing α (pulling southwards) for this family. Increasing α moves the lumps towards s_0 where they coalesce, form a shrinking ring structure and pinch off. Decreasing α has the same effect, except the ring is centred on 0 rather than s_0. In fact, the solution $\alpha\wp(z)$ is identical, up to the rotation and translation $(U_0, s_0) \in G_0$, to the solution $\wp(z)/(e_1^2 \alpha)$.

When α is close to $1/e_1$ and $|\rho|$ is small but nonzero, the behaviour of $\mathcal{E}(z)$ is intermediate between the two cases described above. It has four peaks, but two of these are larger than the other two (figure 5).

5 The metric on M_2

The argument of Belavin and Polyakov (7) shows that M_2 is the flat valley bottom of Q_2, on which V attains its topological minimum value, 4π. Any departure from M_2 involves increasing V, and hence climbing the valley walls. Consider the initial value problem where W starts on M_2 and is given a small push tangential to it. Then, by energy conservation, it must stay close to M_2 during its subsequent evolution. In the geodesic approximation one constrains the configuration to lie on M_2 for all time, but allows the position in M_2 to evolve in time according to the constrained action principle. Since $V = 4\pi$ always, the dynamics is determined solely by the kinetic energy functional (4). Using the homeomorphism ϕ we can transfer the differentiable structure of G/G_0 to M_2. Let $\{q_i : i = 1, 2, \ldots, 8\}$ be local coordinates on M_2, and consider the kinetic energy T as q^i vary in time:

$$T = g_{ij}(q)\dot{q}^i\dot{q}^j \tag{42}$$

where

$$g_{ij}(q) = \mathrm{Re}\int_{T^2} \frac{1}{(1+|W|^2)^2}\frac{\partial W}{\partial q_i}\frac{\partial \overline{W}}{\partial q^j}. \tag{43}$$

Equation (43) defines a Riemannian metric on M_2, $g = g_{ij}dq^i dq^j$, and furthermore the constrained Euler-Lagrange equation (obtained by varying the action $S[q] = \int dt\, T(q, \dot{q})$) is the geodesic equation for (M_2, g). The conjecture is, then, that geodesics in this Riemannian manifold are, when travelled at low speed, close to low-energy two-lump dynamical solutions of the $\mathbb{C}P^1$ model. Some justification for this can be found when comparison is made with other models for which the approximation has been used. In the case of abelian-Higgs vortices, for example, the approximation [3] is supported by rigorous analysis [20] and extensive numerical solution of the full field equations [21].

Ideally, one would like an explicit, closed-form expression for the metric g, but this is rarely possible in practice. There are exceptions [2, 6, 22], but unfortunately this is not one of them. It is possible to place fairly strong constraints on the possible form of g, but not to write it down explicitly (to do so requires, naively, the evaluation of 36 integrals over T^2, each with 8 parameters). It is convenient to lift the geometry to the covering space $(G, \widetilde{g})$ where $\widetilde{g} = \widetilde{\phi}^* g$, the pullback of the metric g by the covering projection $\widetilde{\phi}$. The most useful constraint is that $\widetilde{g}$ is a product metric $\widetilde{g} = \widehat{g} \oplus \delta$ on $PSL(2,\mathbb{C}) \times T^2$ where $\delta = 2\pi ds\, d\bar{s}$. By product metric [23] we mean block diagonal with $\widehat{g}$ independent of position in T^2 and δ independent of position in $PSL(2,\mathbb{C})$.

This is easily established if we recall that any $W \in M_2$ is a rational function of $\wp(z-s)$, so denoting by μ any one of the six $PSL(2,\mathbb{C})$ moduli, we see that

$$\begin{aligned} \frac{\partial W}{\partial s} &= -R_1(\wp(z-s))\wp'(z-s) \\ \frac{\partial W}{\partial \mu} &= R_2(\wp(z-s)) \end{aligned} \tag{44}$$

where R_1, R_2 are rational functions. So the (μ, s) component of $\widetilde{g}$ is

$$\widetilde{g}_{\mu s} = \mathrm{Re}\int_{T^2} f(\wp(z-s))\wp'(z-s) = \mathrm{Re}\int_{T^2} f(\wp(z))\wp'(z) = 0 \tag{45}$$

since $\wp$ is even while $\wp'$ is odd (here $f(u) = -(1+|R_1(u)|^2)^{-2}\overline{R_1(u)}R_2(u)$). Similarly, $\widetilde{g}_{\mu\bar{s}} = 0$, so $\widetilde{g}$ is block diagonal as claimed. Translation symmetry implies that $\widetilde{g}$, and hence $\widehat{g}$, must be independent of s. Hence, it remains to show that δ is independent of the $PSL(2,\mathbb{C})$ moduli. Let $W(x, y, t) = \widetilde{W}(z - s(t))$, $\widetilde{W} \in M_2$, and compute the kinetic energy,

$$T = \int_{T^2} \frac{|\widetilde{W}'|^2}{(1+|\widetilde{W}|^2)^2}|\dot{s}|^2 = \frac{1}{2}V[\widetilde{W}]|\dot{s}|^2 = 2\pi|\dot{s}|^2. \tag{46}$$

Since this is, by definition, $g_{s\bar{s}}|\dot{s}|^2$, we read off the metric $\delta = 2\pi ds\, d\bar{s}$ on T^2.

The geodesic equation for $(G, \widetilde{g})$ decouples into independent geodesic equations for $(PSL(2,\mathbb{C}), \widehat{g})$ and (T^2, δ). Consequently, we may identify s as an effective "centre of mass coordinate" which drifts on T^2 at constant velocity, independent of the lumps' relative motion in $PSL(2,\mathbb{C})$. Without loss of generality, therefore, we can investigate geodesic motion in the reduced covering space $(\widehat{G}, \widehat{g})$, where $\widehat{G}$ denotes $PSL(2,\mathbb{C})$. So lump dynamics in the geodesic approximation has Galilean boost symmetry. This may be understood as a remnant of the Lorentz symmetry of the $\mathbb{C}P^1$ model in $\mathbb{R}^{2+1}$: the field equation is still Lorentz invariant, but the spatial boundary conditions now are not. Under a Lorentz boost, they suffer Lorentz contraction. In a low speed approximation such as this, however, Galilean boost symmetry is recovered, since the spatial contraction is a high order effect.

One further constraint on $\widehat{g}$ will prove useful: since the kinetic energy functional is invariant under global internal rotations of W (rotations of the codomain S^2), $SU(2)$ acts isometrically by left multiplication on $(\widehat{G}, \widehat{g})$ [6]. Briefly, $\widehat{g}$ (or $\widetilde{g}$) is *left-invariant* under $SU(2)$.

6 Geodesic incompleteness of (M_2, g)

One of the most basic questions one can ask about a Riemannian manifold without boundary is whether it is geodesically complete, that is, whether all geodesics can be extended infinitely in time (forwards and backwards). In view of the noncompactness of M_2, this is a nontrivial question for (M_2, g). We will prove that (M_2, g) is , in fact, geodesically incomplete, by finding a geodesic which, although maximally extended, has only finite length (since geodesics are traversed at constant speed, this is sufficient). This geodesic is obtained explicitly, despite our lack of explicit information about g, by using discrete isometries to identify a one dimensional geodesic submanifold. Such arguments have been used to obtain multimonopole scattering geodesics [24] given similarly scant knowledge of the metric on moduli space.

The key observation is that the fixed point set of a discrete group of isometries of a Riemannian manifold is (if a submanifold) a totally geodesic submanifold, that is, a geodesic which starts on and tangential to the fixed point set must remain on the fixed point set for all subsequent time. This follows directly from uniqueness of solutions to the initial value problem of an ordinary differential equation. If a discrete group is found whose fixed point set is diffeomorphic to $\mathbb{R}$, then the set itself is a geodesic.

The following mappings are isometries of $(\widehat{G}, \widehat{g})$,

$$P : M \quad \mapsto \quad \overline{M} \tag{47}$$

$$R : M \quad \mapsto \quad \tau_3 M \tau_3. \tag{48}$$

To see this, consider their effect on $W(z) = M \odot (\wp(z)/e_1)$,

$$P : W(z) \quad \mapsto \quad \overline{M} \odot \frac{\wp(z)}{e_1} = \overline{M} \odot \frac{\overline{\wp(\bar{z})}}{e_1} = \overline{W(\bar{z})} \tag{49}$$

$$R : W(z) \quad \mapsto \quad (\tau_3 M \tau_3) \odot \frac{\wp(z)}{e_1} = -[M \odot (-\wp(z)/e_1)] = -W(iz). \tag{50}$$

So P produces simultaneous reflexions in both domain ($z \mapsto \bar{z}$) and codomain ($W \mapsto \overline{W}$), while R produces rotations of $\pi/2$ in the domain ($z \mapsto iz$) and π in the codomain ($W \mapsto -W$), all of which are symmetries of the $\mathbb{C}P^1$ model. The composition of P and R in either order (they commute) is another isometry of $(\widehat{G}, \widehat{g})$, Since $P^2 = R^2 = (PR)^2 = Id$, the isometries $\{Id, P, R, PR\}$ form the Viergruppe V_4 under composition. A straightforward calculation shows that $\widehat{\Sigma}_V$, the fixed point set of V_4 is

$$\widehat{\Sigma}_V = \{\mathrm{diag}((\alpha e_1)^{\frac{1}{2}}, (\alpha e_1)^{-\frac{1}{2}}) : \alpha \in \mathbb{R}^+\}. \tag{51}$$

This is clearly a submanifold of $\widehat{G}$ diffeomorphic to $\mathbb{R}$, and hence is a geodesic. Its image under the projection $\widetilde{\phi}$ is $\Sigma_V = \{\alpha\wp(z) : \alpha \in \mathbb{R}^+\}$ which is a geodesic of (M_2, g), also diffeomorphic to $\mathbb{R}$.

The submanifold Σ_V was described at the end of section 4. The lump motion corresponding to this geodesic is an infinitely tall thin ring centred at 0 in the past spreading out into four distinct identical peaks, which recombine to form an infinitely tall thin ring centred on s_0 in the future (assuming that Σ_V is traversed in the sense of increasing α). The question remains whether Σ_V is traversed in finite time, i.e. has finite length, and to answer this one needs to understand the induced metric g_V on Σ_V. The restriction of g to Σ_V is

$$g_V = f(\alpha)d\alpha^2 \tag{52}$$

where

$$f(\alpha) = \int_{T^2} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2}. \tag{53}$$

Note that f is clearly positive and decreasing, and is easily shown to have limits ∞ and 0 as $\alpha \to 0$ and $\alpha \to \infty$ respectively. To prove that Σ_V has finite length we will need detailed asymptotic estimates for f in these two limits. The identity (17) implies that

$$f(\alpha) \equiv \frac{1}{(\alpha e_1)^4} f(\frac{1}{\alpha e_1^2}), \tag{54}$$

so the behaviour in one limit follows directly from the behaviour in the other.

Lemma 2 *The following asymptotic formulae hold,*

$$f(\alpha) \sim \frac{\pi^2}{4\alpha} \qquad \text{as } \alpha \to 0 \tag{55}$$

$$f(\alpha) \sim \frac{\pi^2}{4e_1^2\alpha^3} \qquad \text{as } \alpha \to \infty. \tag{56}$$

Proof: We need only prove (55) since (56) follows from this and equation (54). The idea is to split the integration region of (53) into a small neighbourhood of 0 and its complement, bound the contribution of the latter region and use a Laurent expansion in the former.

Fix some $\epsilon \in (0, \frac{1}{4})$ and split T^2 into $D_\epsilon(0) \coprod (T^2 \backslash D_\epsilon(0))$, where $D_\epsilon(0)$ is the open disk of radius ϵ centred on 0. Then

$$\int_{D_\epsilon(0)} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2} < f(\alpha) < \int_{D_\epsilon(0)} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2} + \int_{T^2\backslash D_\epsilon(0)} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2} \tag{57}$$

and $|\wp|$ is bounded on $T^2 \backslash D_\epsilon(0)$, so there exists $M_\epsilon \in (0, \infty)$ such that

$$\int_{T^2\backslash D_\epsilon(0)} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2} < \int_{T^2\backslash D_\epsilon(0)} |\wp|^2 < M_\epsilon \tag{58}$$

independent of α. Hence

$$\int_{D_\epsilon(0)} \frac{\alpha|\wp|^2}{(1+\alpha^2|\wp|^2)^2} < \alpha f(\alpha) < \alpha M_\epsilon + \int_{D_\epsilon(0)} \frac{\alpha|\wp|^2}{(1+\alpha^2|\wp|^2)^2}, \tag{59}$$

so it suffices to prove that

$$\lim_{\alpha\to 0} \int_{D_\epsilon(0)} \frac{\alpha|\wp|^2}{(1+\alpha^2|\wp|^2)^2} = \frac{\pi^2}{4}. \tag{60}$$

The function $h(z) = z^2\wp(z)$ is analytic, bounded and (since $s_0 \notin D_\epsilon(0)$) bounded away from 0 on $D_\epsilon(0)$. So $\wp(z) = h(z)/z^2$ where $0 < c_1 < |h(z)| < c_2 < \infty$, c_1 and c_2 being constants. Defining $\gamma = \epsilon/\sqrt{\alpha}$ and $u = z/\sqrt{\alpha}$,

$$\begin{aligned}\lim_{\alpha\to 0}\int_{D_\epsilon(0)} \frac{\alpha|\wp|^2}{(1+\alpha^2|\wp|^2)^2} &= \lim_{\alpha\to 0}\int_{D_\epsilon(0)} dz\, d\bar{z}\, \frac{\alpha|h(z)|^2/|z|^4}{(1+\alpha^2|h(z)|^2/|z|^4)^2} \\ &= \lim_{\alpha\to 0}\int_{\mathbb{C}} du\, d\bar{u}\, \chi_\gamma(u)\frac{|h(\sqrt{\alpha}u)|^2|u|^4}{(|u|^4+|h(\sqrt{\alpha}u)|^2)^2} \qquad (61)\end{aligned}$$

where χ_γ is the characteristic function of the disk (i.e. $\chi_\gamma(u) = 1$ if $|u| < \gamma$, 0 otherwise). The integrand of (61) is bounded above, independent of α, by

$$\frac{c_2^2|u|^4}{(c_1^2+|u|^4)^2} \qquad (62)$$

which is integrable on $\mathbb{C}$. Hence, Lebesgue's dominated convergence theorem applies [25], and we may interchange the order of limit and integration in equation (61). From the Laurent expansion of $\wp$ about 0,

$$\wp(z) = \frac{1}{z^2} + O(z^2) \qquad (63)$$

one sees that $h(z) = 1 + O(z^4)$, whence

$$\lim_{\alpha\to 0}\frac{\chi_\gamma(u)|h(\sqrt{\alpha}u)|^2|u|^4}{(|u|^4+|h(\sqrt{\alpha}u)|^2)^2} = \frac{|u|^4}{(1+|u|^4)^2}. \qquad (64)$$

Integrating this function over $\mathbb{C}$ yields $\frac{\pi}{4}$, which completes the proof. □

There now immediately follows

Theorem 2 *The moduli space (M_2, g) is geodesically incomplete.*

Proof: We need only prove that the length of Σ_V,

$$l = \int_0^\infty d\alpha\, \sqrt{f(\alpha)} \qquad (65)$$

is finite. By Lemma 2 there exist $0 < \alpha_1 < \alpha_2 < \infty$, $0 < c_3, c_4 < \infty$ such that $f(\alpha) < c_3/\alpha$ on $(0, \alpha_1)$ and $f(\alpha) < c_4/\alpha^3$ on (α_2, ∞). Hence,

$$\begin{aligned} l &< 2\sqrt{c_3\alpha_1} + \int_{\alpha_1}^{\alpha_2} d\alpha\, \sqrt{f(\alpha)} + 2\sqrt{\frac{c_4}{\alpha}} \\ &< 2\sqrt{c_3\alpha_1} + (\alpha_2-\alpha_1)f(\alpha_1) + 2\sqrt{\frac{c_4}{\alpha}} \qquad (66)\end{aligned}$$

by monotonicity of f. □

The geodesic approximation predicts, then, that lumps (at least when coincident) can shrink and form singularities in finite time. Shrinking has been observed in numerical simulations of the $\mathbb{C}P^1$ model in both the plane [26] and the torus [27], although the particular initial value problem considered here has not been simulated.

7 A two-dimensional geodesic submanifold

The Viergruppe V_4 has three $\mathbb{Z}_2$ subgroups $\{Id, P\}$, $\{Id, R\}$ and $\{Id, PR\}$ whose fixed point sets $\widehat{\Sigma}_P$, $\widehat{\Sigma}_R$, $\widehat{\Sigma}_{PR}$ respectively are all geodesic submanifolds of $(\widehat{G}, \widehat{g})$. Of these, $\widehat{\Sigma}_R$ is two dimensional (the others are three dimensional) and projects under $\widetilde{\phi}$ to

$$\Sigma_R = \{\alpha e^{i\psi}\wp(z) : \alpha \in \mathbb{R}^+, \psi \in [0, 2\pi]\}, \qquad (67)$$

a geodesic submanifold of (M_2, g) diffeomorphic to a cylinder. This has a tractable geodesic problem. Recalling that g is left-invariant under $SU(2)$, its restriction g_R to Σ_R is independent of ψ. In fact,

$$g_R = f(\alpha)(d\alpha^2 + \alpha^2 d\psi^2) \tag{68}$$

where f is the same function defined in (53).

Lemma 2 implies that the asymptotic form of g_R towards the ends of the cylinder is

$$g_R \sim \frac{\pi^2}{4\alpha}(d\alpha^2 + \alpha^2 d\psi^2) \qquad \text{as } \alpha \to 0 \tag{69}$$

$$g_R \sim \frac{\pi^2}{4e_1^2\alpha^3}(d\alpha^2 + \alpha^2 d\psi^2) \qquad \text{as } \alpha \to \infty. \tag{70}$$

The formula in (69) is the metric of a flat, singular cone with deficit angle π, so (Σ_R, g_R) can be visualized as having a conical singularity at $\alpha = 0$. By virtue of the identity (54), the metric g_R is invariant under the mapping $\alpha \mapsto (e_1^2\alpha)^{-1}$, and consequently (Σ_R, g_R) has an identical conical singularity at $\alpha = \infty$, as may be shown by the reparametrization $\beta = (e_1^2\alpha)^{-1}$ in (70). So (Σ_R, g_R) is a rotationally symmetric cylinder of finite length with its ends pinched to identical cones. It is the internal rotation orbit, about a fixed axis, of the one parameter family of exceptionally symmetric configurations already described (Σ_V). The singular points $\alpha = 0$ and $\alpha = \infty$ correspond to infinitely narrow, spiky, ring like configurations centred on 0 and s_0 respectively. Motion on Σ_R corresponds to rotational and shape changing motion of the double lump on the torus. The conserved kinetic energy of this motion is

$$T = f(\alpha)\dot{\alpha}^2 + \frac{J^2}{4\alpha^2 f(\alpha)}, \tag{71}$$

where $J = \alpha^2 f(\alpha)\dot{\psi}$ is the conserved angular momentum conjugate to ψ. One may imagine the dynamics as that of a point particle moving on the interval $(0, \infty)$ with position dependent mass and subject to a potential. Geodesic motion is invariant under rescaling of time, so one can restrict attention to the two cases $J^2 = 0$ and $J^2 = 1$. If $J^2 = 0$ the motion is irrotational and the point particle travels from one conical singularity to the other in finite time along a path of constant ψ. These geodesics are just rotated versions of Σ_V. The more interesting case is when $J^2 = 1$ where the nature of the motion is determined by the centrifugal potential

$$\mathcal{U}(\alpha) = \frac{1}{4\alpha^2 f(\alpha)}. \tag{72}$$

From the asymptotic formulae of proposition 1 we see that the potential has the asymptotic behaviour

$$\mathcal{U}(\alpha) \sim \frac{}{\pi^2\alpha}, \quad \text{as } \alpha \to 0 \tag{73}$$

$$\mathcal{U}(\alpha) \sim \frac{e_1^2}{\pi^2}\alpha \quad \text{as } \alpha \to \infty \tag{74}$$

implying that $\mathcal{U}$ must have at least one stable equilibrium. The identity (54) implies a similar identity for the potential,

$$\mathcal{U}(\frac{1}{\alpha e_1^2}) \equiv \mathcal{U}(\alpha). \tag{75}$$

Differentiating both sides of (75) one finds that $\mathcal{U}$ has a critical point at $\alpha = 1/e_1$, the fixed point of the isometry $\alpha \mapsto (\alpha e_1^2)^{-1}$. Numerical evaluation of $\mathcal{U}$ suggests that this is the only critical point, a global minimum, so that $\mathcal{U}$ is a single potential well (see figure 6). Since $\mathcal{U}$ grows unbounded as $\alpha \to 0$ and $\alpha \to \infty$, all motion in the well is oscillatory. So these geodesics wind around Σ_R, passing back and forth along its length indefinitely. They are bounded away from the singularities by angular momentum conservation. They correspond to rotational motions of the double lump during which the arrows of the configuration spin about the north-south axis of S^2, and its shape periodically oscillates about that of the most symmetric configuration, $W(z) = \wp(z)/e_1$.

8 The diameter of (M_2, g)

In this section we will prove that (M_2, g) has finite size, in an appropriate sense. Since one is interested in (M_2, g) primarily for its geodesics, a linear measure of size is most meaningful, so we will consider its diameter. Recall that (M_2, g), like any Riemannian manifold, has a natural distance function $d : M_2 \times M_2 \to \mathbb{R}$ where $d(W, W')$ is the infimum of lengths with respect to g of piecewise C^1 paths in M_2 connecting W and W' (note that d has nothing to do with D, the distance function defined in section 3, although they define equivalent topologies on M_2). The diameter of (M_2, g) is simply the diameter of the associated metric space (M_2, d), that is,

$$\mathrm{diam}(M_2, g) = \sup_{W, W' \in M_2} d(W, W'). \tag{76}$$

Once again, it is the noncompactness of M_2 which makes this diameter interesting, and its finiteness nontrivial. The geometric meaning of the result is that all points lie within a bounded distance of each other, and, in particular, no point lies far from the end of M_2, where the static solutions collapse to singular, spiky configurations. Thus all static solutions are close to collapse in this geometry. This may be the underlying cause of the ubiquitous instability found in numerical simulations of two-lump scattering on the torus [27].

Theorem 3 *The moduli space (M_2, g) has finite diameter.*

Proof: It suffices to prove that the covering space $(G, \widetilde{g})$ has finite diameter and, further, since $\widetilde{g} = \widehat{g} \oplus \delta$ is a product metric and T^2 is compact, it is sufficient to prove that the reduced covering space $(\widehat{G}, \widehat{g})$ has finite diameter. By the triangle inequality for $d : \widehat{G} \times \widehat{G} \to \mathbb{R}$,

$$\mathrm{diam}(\widehat{G}, \widehat{g}) \leq 2 \sup_{W \in \widehat{G}} d(W, W_0), \tag{77}$$

where W_0 is any point in $\widehat{G}$. Let $W_0 = \wp$. We will explicitly construct a path from $W \in \widehat{G} \cong SO(3) \times [\mathbb{R}^+ \times \mathbb{C}]$ to W_0, and bound its length independent of W.

Let $W = U \odot [\alpha(\wp + \rho)]$. The first piece of the path has $(\alpha, \rho) \in \mathbb{R}^+ \times \mathbb{C}$ fixed, but takes U to I. For example, since any $U \in SU(2)$ is $\exp(u)$ for some $u \in su(2)$, we could consider the path $\Omega(t) = \exp((1-t)u)$, so that $\Omega(0) = U$ while $\Omega(1) = I$. Denote by $\gamma(\alpha, \rho)$ the metric on $SO(3)$ induced by $\widehat{g}$ at fixed (α, ρ). Since $SO(3)$ is compact the length of $\Omega(t)$ is bounded independent of U for each (α, ρ). One must check, however, that the length remains bounded as a function of (α, ρ). Since $\gamma(\alpha, \rho)$ is a left-invariant metric on $SO(3)$, it suffices to show that

$$\Gamma(\alpha, \rho) := \sum_{i,j=1}^{3} |\gamma_{ij}(\alpha, \rho)| \tag{78}$$

is a bounded function, where $\gamma_{ij}(\alpha, \rho)$ are the metric coefficients of γ evaluated at $I \in SO(3)$ with respect to a particular choice of basis for $T_I SO(3)$. The basis used does not matter. One convenient choice consists of the three vectors represented by the curves

$$\exp\left(it\frac{\tau_i}{2}\right) \qquad i = 1, 2, 3 \qquad t \in (-\epsilon, \epsilon), \tag{79}$$

where τ_i are the Pauli matrices (this is equivalent to choosing $\{i\tau_i/2 : i = 1, 2, 3\}$ as a basis for $su(2)$). Elementary calculation then shows that $\Gamma < 3$ for all (α, ρ). For example, $\gamma_{33}(\alpha, \rho)$ is the squared length of the vector $[\exp(it\tau_3/2)]$. Let $w := \alpha(\wp + \rho)$. Then

$$\begin{aligned} W(z, t) &= \exp\left(it\frac{\tau_3}{2}\right) \odot w(z) = e^{it} w(z) \\ \dot{W}(z) &= \left.\frac{\partial W}{\partial t}\right|_{t=0} = iw(z) \end{aligned} \tag{80}$$

and

$$\gamma_{33}(\alpha,\rho) = \int_{T^2} \frac{|\dot{W}|^2}{(1+|W|^2)^2} = \int_{T^2} \frac{|w|^2}{(1+|w|^2)^2} < \frac{1}{2}. \tag{81}$$

Bounds on the other metric coefficients are equally straightforward.

It remains to construct a path from $\alpha(\wp+\rho)$ to $\wp$ with length bounded above independent of (α,ρ). It is necessary to split $\mathbb{R}^+ \times \mathbb{C}$ into two pieces $X_+ \coprod X_-$ and construct the path differently in each piece. Here $X_+ = \{(\alpha,\rho) \in \mathbb{R}^+ \times \mathbb{C} : \alpha > 1\}$ and X_- is its complement.

For any $(\alpha,\rho) \in X_-$ construct the path $x_- : [0,1] \to \mathbb{R}^+ \times \mathbb{C}$ where

$$x_-(t) = \left\{ \begin{array}{cc} (\alpha, (1-2t)\rho) & t \in [0, \frac{1}{2}] \\ (1 + 2(1-\alpha)(t-1), 0) & t \in (\frac{1}{2}, 1] \end{array} \right. \tag{82}$$

so that $x_-(0) = (\alpha,\rho)$, $x_-(1) = (1,0)$. Thinking of $\mathbb{R}^+ \times \mathbb{C}$ as the upper half of $\mathbb{R}^3$, this path consists of a horizontal line from (α,ρ) to $(\alpha,0)$ followed by a vertical line from $(\alpha,0)$ to $(1,0)$ (see figure 7). Its length is bounded above by the sum of the lengths of the curves $\{(\alpha, te^{i\psi}) : t \in [0,\infty)$ and $\{(t,0) : t \in (0,1]\}$, where $\psi = \arg\rho$. So

$$l[x_-] < l_1(\alpha,\psi) + l_3 \tag{83}$$

where

$$\begin{aligned} l_1(\alpha,\rho) &= \int_0^\infty d|\rho| \sqrt{\widehat{g}_{|\rho||\rho|}(\alpha,\rho)} = \int_0^\infty d|\rho| \left[\int_{T^2} \frac{\alpha^2}{(1+\alpha^2|\wp + |\rho| e^{i\psi}|^2)^2} \right]^{\frac{1}{2}} \\ l_3 &= \int_0^1 d\alpha \sqrt{\widehat{g}_{\alpha\alpha}(\alpha,0)} = \int_0^1 d\alpha \left[\int_{T^2} \frac{|\wp|^2}{(1+\alpha^2|\wp|^2)^2} \right]^{\frac{1}{2}}. \end{aligned} \tag{84}$$

That l_3 is finite follows directly from Lemma 2, since $\widehat{g}_{\alpha\alpha}(\alpha,0)$ is precisely $f(\alpha)$, the function previously discussed. To prove that $l_1(\alpha,\rho)$ is finite and bounded independent of $(\alpha,\psi) \in (0,1] \times [0,2\pi]$ is more involved. By a change of variable, $\sigma := \alpha|\rho|$, we can rewrite $l_1(\alpha,\psi)$ as

$$l_1(\alpha,\psi) = \int_0^\infty d\sigma \left[\int_{T^2} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} \right]^{\frac{1}{2}}. \tag{85}$$

One must now appeal to a technical lemma, whose proof we postpone:

Lemma 3 *There exist $\sigma_*, C > 0$, independent of (α,ψ), such that $\forall \sigma > \sigma_*$ and $\alpha \leq 1$,*

$$\int_{T^2} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} < \frac{C}{\sigma^3}. \qquad \clubsuit \tag{86}$$

It follows that

$$\begin{aligned} l_1(\alpha,\psi) &= \int_0^{\sigma_*} \left[\int_{T^2} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} \right]^{\frac{1}{2}} + \int_{\sigma_*}^\infty \left[\int_{T^2} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} \right]^{\frac{1}{2}} \\ &< \sigma_* + \int_{\sigma_*}^\infty \sqrt{\frac{C}{\sigma^3}} < C' < \infty \end{aligned} \tag{87}$$

C' being a constant.

Now, for any $(\alpha,\rho) \in X_+$ construct the path $x_+ : [0,1] \to \mathbb{R}^+ \times \mathbb{C}$, where

$$x_+(t) = \left\{ \begin{array}{cc} (\alpha - 2(\alpha-1)t, \rho) & t \in [0, \frac{1}{2}] \\ (1, 2(1-t)\rho) & t \in (\frac{1}{2}, 1], \end{array} \right. \tag{88}$$

consisting (see figure 7) of a vertical line from (α, ρ) to $(1, \rho)$ followed by a horizontal line from $(1, \rho)$ to $(1, 0)$. Its length is bounded above by the sum of the lengths of the lines $\{(t, \rho) : t \in [1, \infty)\}$ and $\{(1, te^{i\psi}) : t \in [0, \infty)\}$. So

$$l[x_+] < l_2(\rho) + l_1(1, \psi) \tag{89}$$

where l_1 was previously defined and

$$l_2(\rho) = \int_1^\infty d\alpha \sqrt{\widehat{g}_{\alpha\alpha}(\alpha, \rho)} = \int_1^\infty d\alpha \left[\int_{T^2} \frac{|\wp + \rho|^2}{(1 + \alpha^2 |\wp + \rho|^2)^2} \right]^{\frac{1}{2}}. \tag{90}$$

We have already shown that $l_1(1, \psi)$ is finite and bounded independent of ψ (this follows from Lemma 3 in the case $\alpha = 1$).

That $l_2(\rho)$ is finite $\forall \rho \in \mathbb{C}$ is easily shown, using an argument similar to that of Lemma 2. Let z_1, z_2 be the roots of $\wp + \rho$ (possibly coincident) and split T^2 into small neighbourhoods of these roots and their complement. In the complement use the trivial bound $|\wp + \rho| \geq C$, constant, while near the roots use Laurent expansions of $\wp + \rho$. One finds that $\widehat{g}_{\alpha\alpha} < C'/\alpha^3$ which is sufficient for finiteness of $l_2(\rho)$ for all ρ, and boundedness of l_2 on any compact subset of $\mathbb{C}$. This is insufficient for our purposes, since $l_2(\rho)$ could grow unbounded as $|\rho| \to \infty$. We again appeal to a technical lemma whose proof we postpone:

Lemma 4 *For all $\rho \in \mathbb{C}$ such that $|\rho| > e_1 + 2$,*

$$\int_{T^2} \frac{|\wp + \rho|^2}{(1 + \alpha^2 |\wp + \rho|^2)^2} < \frac{2}{\alpha^4} + \frac{\pi}{2\alpha^4} \log(1 + \alpha^2). \qquad \clubsuit \tag{91}$$

So for all ρ outside the closed disk $D_{e_1+2}(0)$,

$$l_2(\rho) < \int_1^\infty d\alpha \left[\frac{2}{\alpha^4} + \frac{\pi}{2\alpha^4} \log(1 + \alpha^2) \right]^{\frac{1}{2}} = C < \infty \tag{92}$$

C being a constant. Hence l_2 is bounded independent of ρ, and all points in $\widehat{G}$ lie within a bounded distance of $\wp$, and hence, one another. $\Box$

Proof of Lemma 4: Let $\rho \in \mathbb{C}$ such that $|\rho| > e_1 + 2$. Since $\wp$ is an even function,

$$\widehat{g}_{\alpha\alpha}(\alpha, \rho) = 2 \int_H \frac{|\wp + \rho|^2}{(1 + \alpha^2 |\wp + \rho|^2)^2} \tag{93}$$

where $H = [0, 1) \times [0, \frac{1}{2})$ is the "half torus" (the point is that $\wp$ is injective on H). Split H into two pieces $H_+ \coprod H_-$ where $H_+ = \{z \in H : |\wp + \rho| > 1\}$. Now,

$$\int_{H_+} \frac{|\wp + \rho|^2}{(1 + \alpha^2 |\wp + \rho|^2)^2} < \int_{H_+} \frac{1}{\alpha^4 |\wp + \rho|^2} < \int_{H_+} \frac{1}{\alpha^4} < \frac{1}{\alpha^4}. \tag{94}$$

To estimate the contribution of the H_- region, we perform a variable change $z \mapsto u = \wp(z)$ on H_-. Since $\wp$ is injective on H_-, this variable change is well defined provided $\wp$ has no critical (i.e. double valency) points in H_-. The transformed integration range $\wp(H_-)$ is a closed disk of unit radius centred on $-\rho$, so given that $|\rho| > e_1 + 2$, $\wp(H_-)$ contains none of $\{\infty, 0, \pm e_1\}$, and hence H_- excludes all the double valency points. The Jacobian of the variable change is $|\wp'(z)|^{-2} = |4u(u^2 - e_1^2)|^{-1}$, so

$$\int_{H_-} \frac{|\wp + \rho|^2}{(1 + \alpha^2 |\wp + \rho|^2)^2} = \frac{1}{4} \int_{\wp(H_-)} \frac{du\, d\bar{u}}{|u||u^2 - e_1^2|} \frac{|u + \rho|^2}{(1 + \alpha^2 |u + \rho|^2)^2}. \tag{95}$$

Now, for all $u \in \wp(H_-)$, $|u| \geq e_1 + 1 > 1$, and $|u \pm e_1| \geq ||u| - e_1| \geq 1$, so

$$\begin{aligned}
\int_{H_-} \frac{|\wp + \rho|^2}{(1+\alpha^2|\wp+\rho|^2)^2} &< \frac{1}{4}\int_{\wp(H_-)} du\, d\bar{u}\, \frac{|u+\rho|^2}{(1+\alpha^2|u+\rho|^2)^2} \\
&= \frac{\pi}{2}\int_0^1 dx\, \frac{x^3}{(1+\alpha^2 x^2)^2} \qquad (x := |u+\rho|) \\
&= \frac{\pi}{4\alpha^4}\int_1^{1+\alpha^2} dy\, \frac{y-1}{y^2} \qquad (y := 1+\alpha^2 x) \\
&< \frac{\pi}{4\alpha^4}\log(1+\alpha^2) \qquad (96)
\end{aligned}$$

Using inequalities (94) and (96) in equation (93), the result immediately follows. □

Proof of Lemma 3: The idea is similar to the proof of Lemma 4:

$$\int_{T^2} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} = 2\left[\int_{H_+} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} + \int_{H_-} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2}\right] \qquad (97)$$

where again $H_+ \coprod H_- = H$, the half torus, but now

$$H_+ = \{z \in H : |\alpha\wp + \sigma e^{i\psi}| > \sigma^{\frac{3}{4}}\}. \qquad (98)$$

The H_+ integral is trivially bounded by $1/\sigma^3$. We make the same variable change $z \mapsto u = \wp(z)$ on H_-. Now $\wp(H_-)$ is a closed disk of radius $\sigma^{\frac{3}{4}}/\alpha$ centred on $\sigma e^{i(\psi+\pi)}/\alpha$. In order that $\wp(H_-)$ contain none of $\{\infty, 0, \pm e_1\}$, it suffices that $\sigma \geq \sigma_c$ where σ_c is the real solution of

$$\sigma_c - \sigma_c^{\frac{3}{4}} = 2e_1. \qquad (99)$$

To see this, note that $\forall u \in \wp(H_-)$,

$$|u| \geq \frac{\sigma - \sigma^{\frac{3}{4}}}{\alpha} \geq \sigma - \sigma^{\frac{3}{4}} \geq \sigma_c - \sigma_c^{\frac{3}{4}} = 2e_1, \qquad (100)$$

where the restriction $\alpha \leq 1$ has been used. So the variable change is well defined provided $\sigma \geq \sigma_c$.

Recall that the Jacobian of the transformation is $|4u(u^2 - e_1^2)|^{-1}$. Now $\forall u \in \wp(H_-)$, $|u| \geq 2e_1$ as shown above. Hence

$$|u|^3 = |u|^2|u| \geq 4e_1^2|u|, \qquad (101)$$

and

$$\begin{aligned}
|u(u^2 - e_1^2)| &\geq ||u|^3 - e_1^2|u|| = |u|^3 - e_1^2|u| \\
&\geq |u|^3 - \frac{1}{4}|u|^3 = \frac{3}{4}|u|^3. \qquad (102)
\end{aligned}$$

Thus,

$$\int_{H_-} \frac{1}{(1+|\alpha\wp + \sigma e^{i\psi}|^2)^2} \leq \frac{1}{3}\int_{\wp(H_-)} \frac{du\, d\bar{u}}{|u|^3}\, \frac{1}{(1+|\alpha u + \sigma e^{i\psi}|^2)^2}. \qquad (103)$$

Now let $\tilde{\sigma}_c$ be the real solution of

$$\tilde{\sigma}_c - \tilde{\sigma}_c^{\frac{3}{4}} = \frac{\tilde{\sigma}_c}{2}, \qquad (104)$$

and define $\sigma_* = \sup\{\sigma_c, \tilde{\sigma}_c\}$. Then, provided $\sigma \geq \sigma_*$, for all $u \in \wp(H_-)$

$$|u| \geq \frac{\sigma - \sigma^{\frac{3}{4}}}{\alpha} \geq \frac{\sigma}{2\alpha}. \qquad (105)$$

This allows one to estimate the the $|u|^{-3}$ part of the integrand of inequality (103), which still holds since $\sigma_* \geq \sigma_c$:

$$\begin{aligned}\int_{H_-} \frac{1}{(1+|\alpha\wp+\sigma e^{i\psi}|^2)^2} &\leq \frac{8\alpha^3}{3\sigma^3}\int_{\wp(H_-)} \frac{du\, d\bar{u}}{(1+|\alpha u+\sigma e^{i\psi}|^2)^2} \\ &= \frac{8\alpha^3}{3\sigma^3}\frac{2\pi}{\alpha^2}\int_0^1 dx\, \frac{x}{(1+x^2)^2} \qquad (x := |\alpha u + \sigma e^{i\psi}|) \\ &= \frac{C\alpha}{\sigma^3} \leq \frac{C}{\sigma^3}. \end{aligned} \tag{106}$$

The result immediately follows. □

9 Conclusion

In this paper we have considered the low-energy dynamics of two $\mathbb{C}P^1$ lumps moving on a torus in the framework of the geodesic approximation. We have proved that the degree 2 moduli space M_2 is homeomorphic to the left coset space G/G_0, where G is the eight-dimensional, noncompact Lie group $PSL(2,\mathbb{C}) \times T^2$ and G_0 is a discrete subgroup of order 4. This result provides a good global parametrization of M_2 with unconstrained parameters, based on the Weierstrass $\wp$ function (this situation should be compared with other studies where M_2 was parametrized using the Weierstrass σ function and constrained parameters [27, 28]), and allows a systematic description of the degree 2 static solutions, some of which display four rather than, as one might expect, two distinct energy peaks. By lifting the metric g on M_2 defined by the kinetic energy to $\widetilde{g}$ on the covering space G, we showed that the dynamics decouples into a trivial "centre of mass" motion and a nontrivial relative motion of the lumps. This reduces the problem to geodesic motion in a 6-dimensional reduced covering space $(\widehat{G}, \widehat{g})$. Two further results were proved concerning the Riemannian geometry of (M_2, g), namely that the moduli space is geodesically incomplete and has finite diameter. These imply that static lumps can collapse to singularities in finite time, and that all static solutions are close to such singularities. In addition, a two dimensional geodesic submanifold was identified, and its geometry and geodesics described in detail.

To make further progress in solving the geodesic problem for (M_2, g) one would need to resort to numerical solution of the geodesic equation. Given the explicit parametrization of M_2, and that the metric components are integrals over a compact, two dimensional domain, such numerical work should be reasonably economical. In particular, there are two 3 dimensional geodesic submanifolds whose geodesic problems would be well suited to numerical study, and which should yield interesting lump dynamics. These are Σ_P and Σ_{PR}, the projected fixed point sets of the isometries $P, PR : \widehat{G} \to \widehat{G}$. Explicitly,

$$\begin{aligned}\Sigma_P &= \{\exp(i\psi\tau_2/2) \odot [\alpha(\wp(z)+\rho_1)] : \psi \in [0,2\pi], \alpha \in \mathbb{R}^+, \rho_1 \in \mathbb{R}\} \\ \Sigma_{PR} &= \{\exp(i\psi\tau_1/2) \odot [\alpha(\wp(z)+i\rho_2)] : \psi \in [0,2\pi], \alpha \in \mathbb{R}^+, \rho_2 \in \mathbb{R}\},\end{aligned} \tag{107}$$

so both are internal rotation orbits, about (different) fixed axes, of the $\alpha(\wp+\rho)$ family, but with ρ real (Σ_P) or purely imaginary (Σ_{PR}). On these submanifolds, therefore, the two lumps, when distinct, are constrained to lie either on the central cross and boundary of the unit square, or its diagonals, respectively. In either case, they can only scatter through 90°. In the case of Σ_P, for example, any geodesic which punctures any of the cylinders $\rho_1 = 0$, $\rho_1 = e_1$, $\rho_1 = -e_1$ at α much greater than $1/e_1$ gives rise to 90° scattering of the lumps. Similarly, any geodesic which punctures the $\rho_2 = 0$ cylinder in Σ_{PR} gives rise to 90° scattering along the diagonals. Both these processes have been observed in numerical simulations of the field equation [27]. To understand the long time behaviour of the geodesics after the scattering event would require detailed numerical work.

Other extensions of the present work would be interesting. One can extend the geodesic incompleteness results proved here for (M_2, g) on T^2 and elsewhere [6] for (M_1, g) on S^2 to the general setting of (M_n, g) for the $\mathbb{C}P^1$ model on an arbitrary compact Riemann surface [29]. It may well be possible to similarly extend our result concerning the finite diameter of moduli space to the general setting. Also, one would expect that (M_2, g) has finite volume, as well as diameter (although neither guarantees the other), and perhaps this can be established by making refined versions of estimates such as those in Lemmata 2, 3 and 4. Finally, it should be emphasised that all our results concern an *approximation* to the field theory. While this has proved remarkably successful in all situations where it has been tested, one would ideally like rigorous analysis to back up physical intuition. Given the singularity of the geometry of moduli space for the planar $\mathbb{C}P^1$ model, the model on the torus provides an ideal starting point for an analysis fashioned after Stuart's work on vortices and monopoles [20].

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Figure captions

Figure 1: The fundamental domain of the Weierstrass $\wp$ function: $\wp$ is real on the solid lines and imaginary on the dashed lines. The four double valency points are marked by circles.

Figure 2: Energy density plots of $W(z) = \wp(z) + \rho$ for various values of ρ. In plot (a) $\rho = 1 - i$, so the roots of W are separate and two lumps form. In plots (b), (c) and (d), $\rho = 0, e_1, -e_1$ respectively so the roots of W coincide. Here the energy distribution is roughly annular, centred on the double valency points s_0, s_2, s_1.

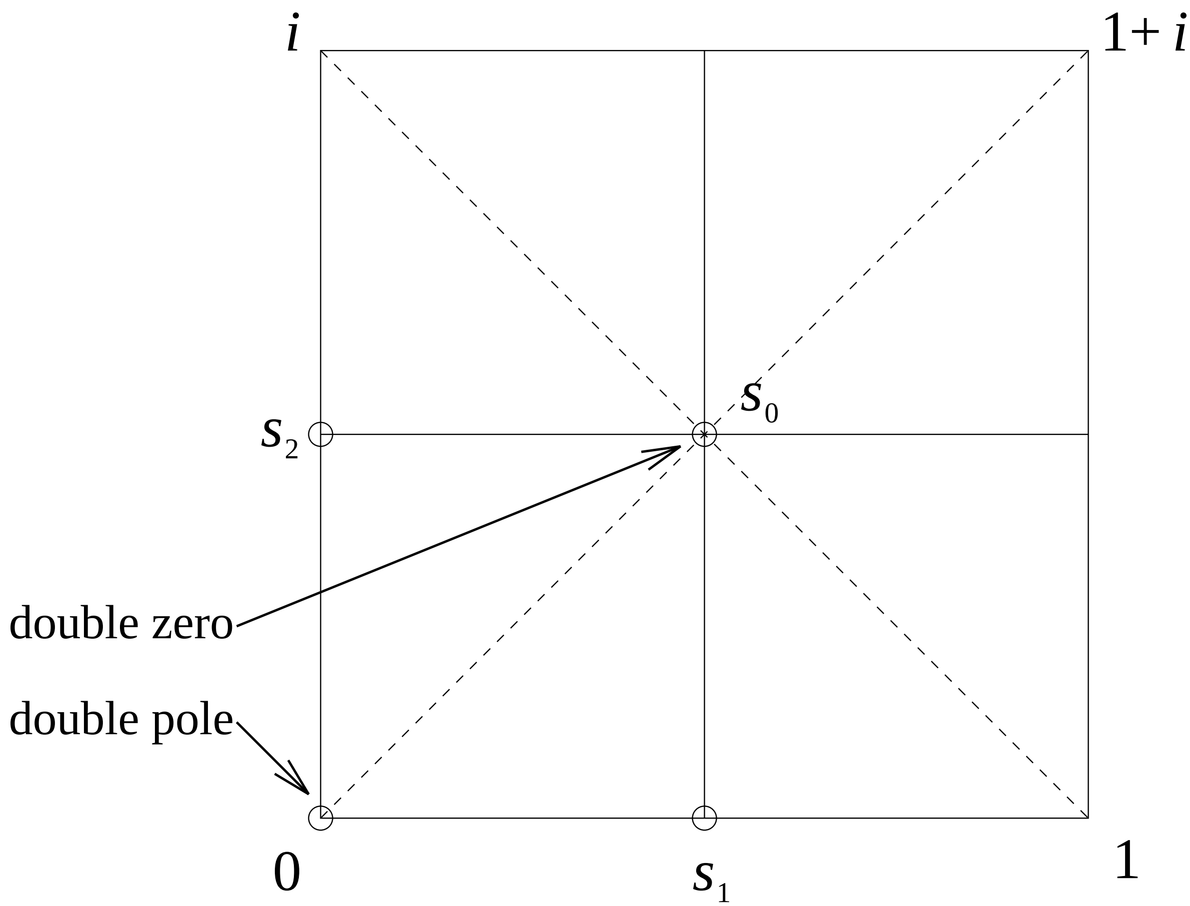
Figure 3: Energy density plots of $W(z) = \alpha(\wp(z) + 1 - i)$ in the cases of (a) large α ($\alpha = 2$) and (b) small α ($\alpha = 0.03$).

Figure 4: The exceptionally symmetric family $W(z) = \alpha\wp(z)$. The parameter values are (a) $\alpha = 4$, (b) $\alpha = 0.3$, (c) $\alpha = 1/e_1$, (d) $\alpha = 0.01$ and (e) $\alpha = 0.005$. Plot (c) depicts the most evenly spread energy distribution possible for a degree 2 static solution.

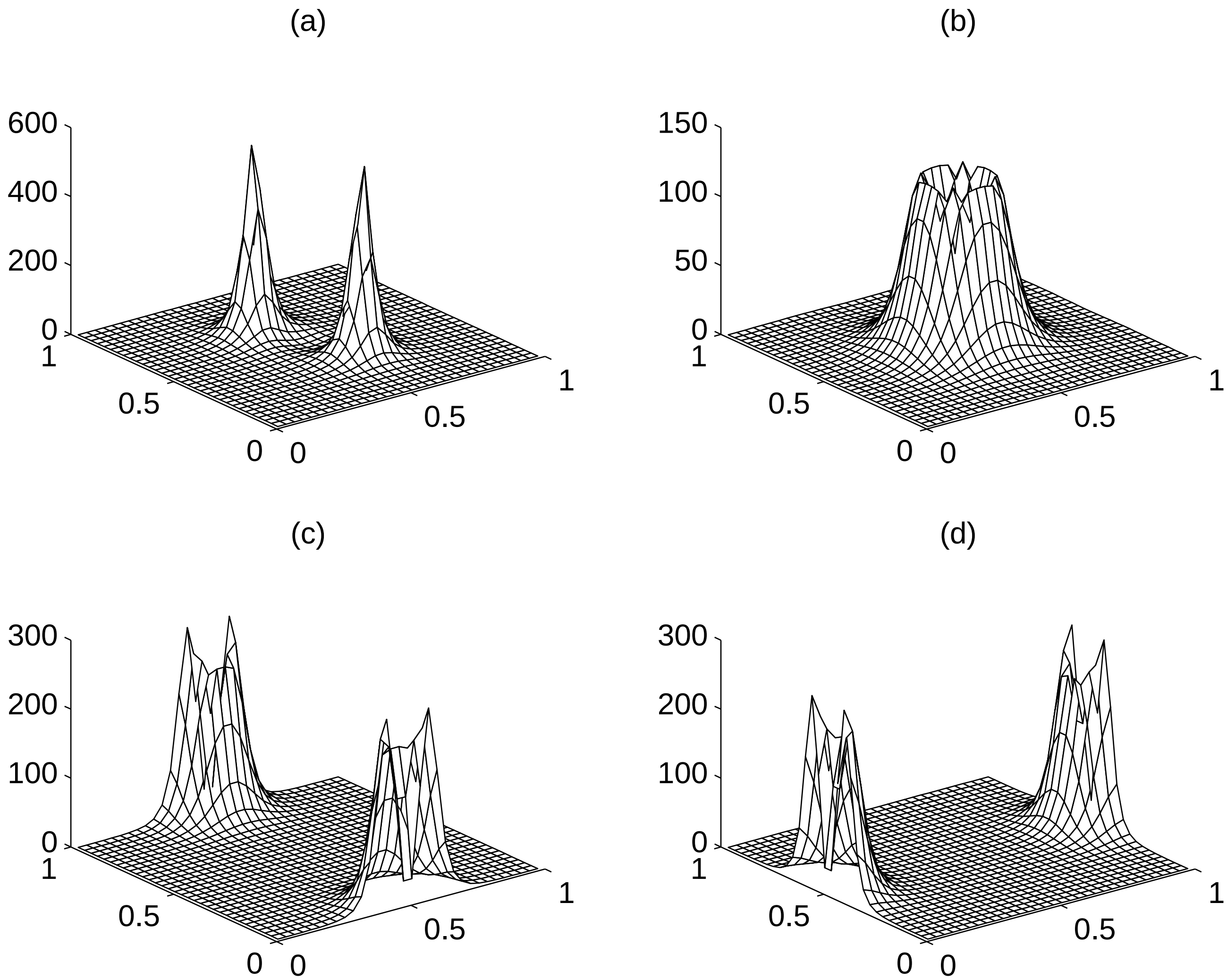
Figure 5: Energy density plot of $W(z) = (\wp(z) - i)/e_1$.

Figure 6: The centrifugal potential $\mathcal{U}(\alpha)$ of equation (72), solid line, compared with the asymptotic formulae for $\mathcal{U}$ for small and large α given in equations (73) and (74), dashed lines.

Figure 7: The paths x_- and x_+ constructed in the proof of Theorem 3.

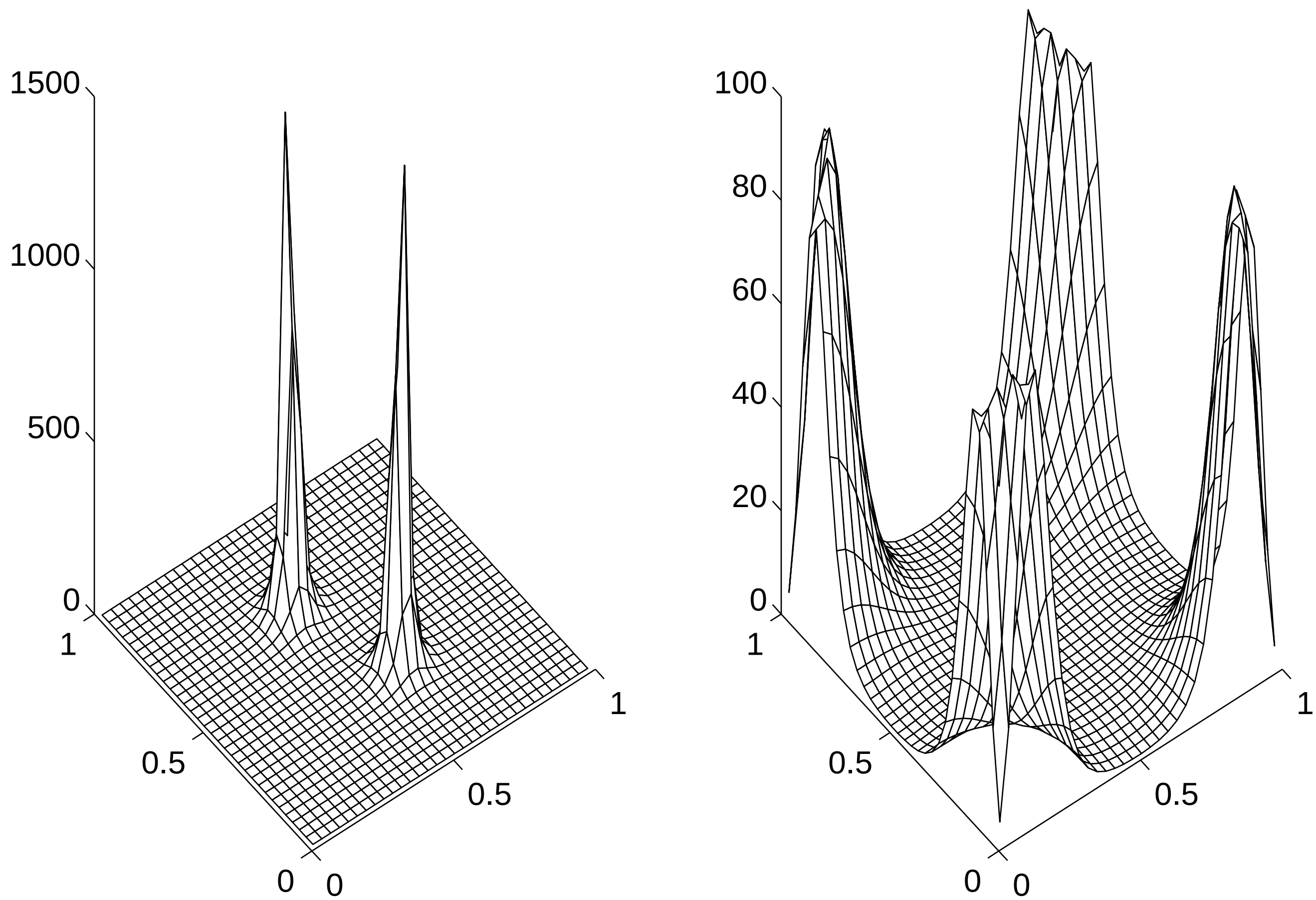


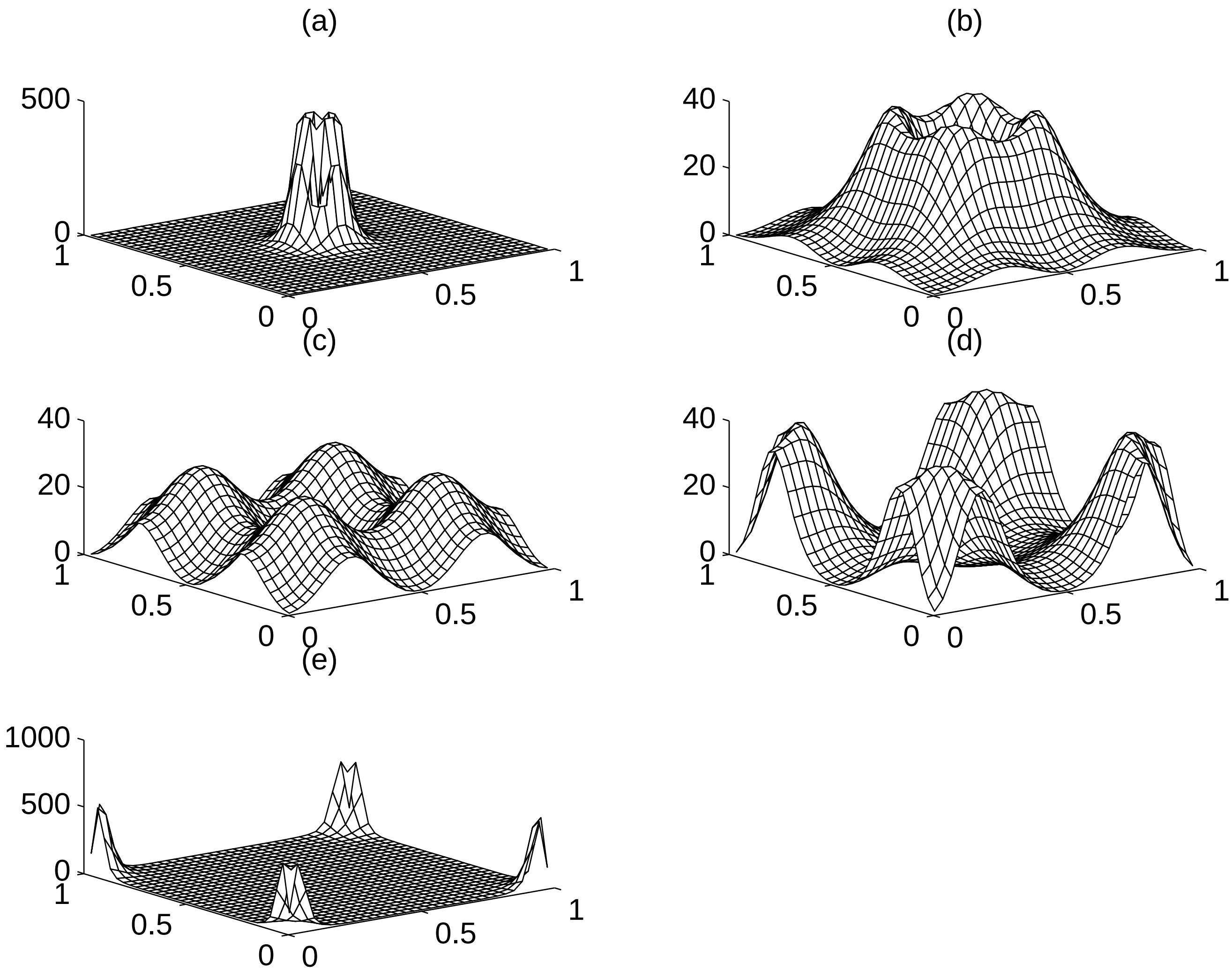
i
$1+i$
s_0
s_2
double zero
double pole
0
s_1
1

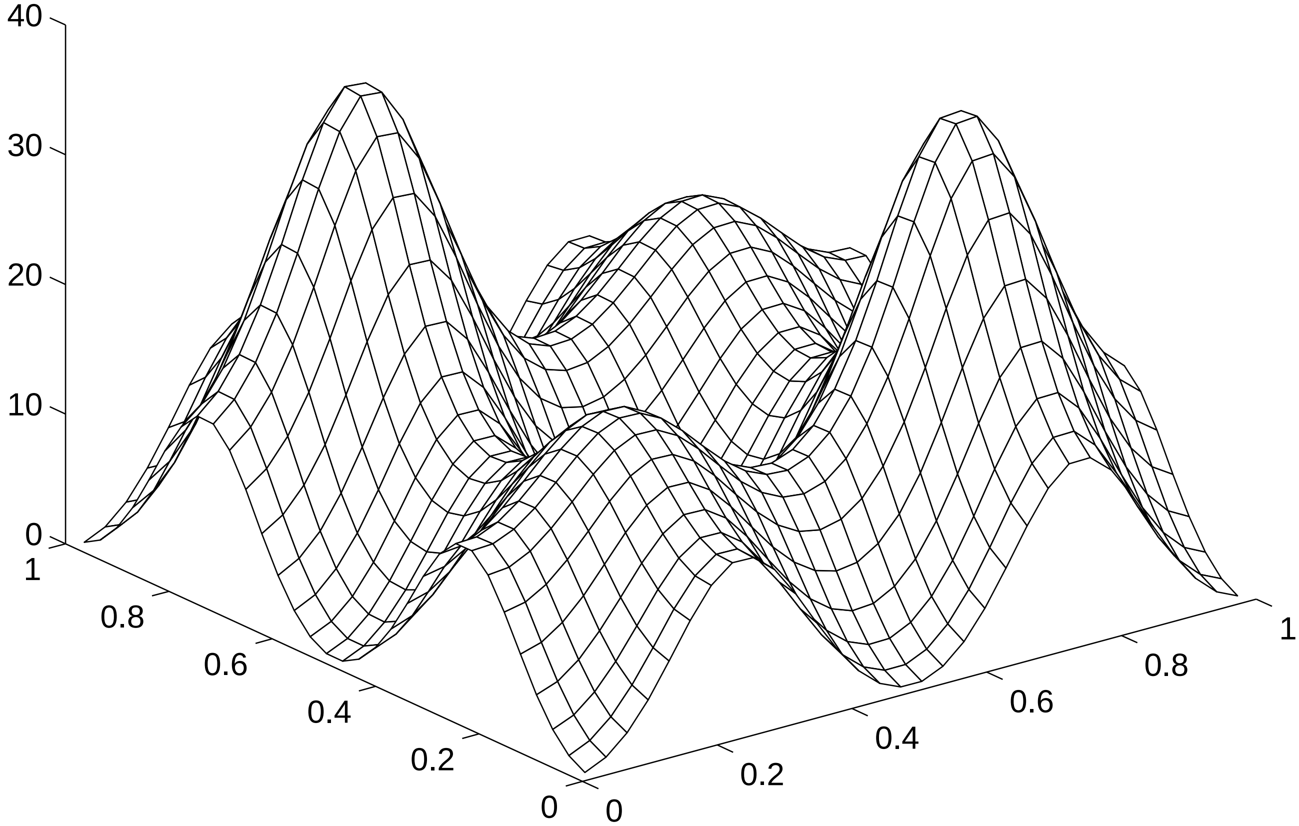


(a)
600
400
200
0
1
0.5
0
0
0.5
1
(b)
150
100
50
0
1
0.5
0
0
0.5
1
(c)
300
200
100
0
1
0.5
0
0
0.5
1
(d)
300
200
100
0
1
0.5
0
0
0.5
1

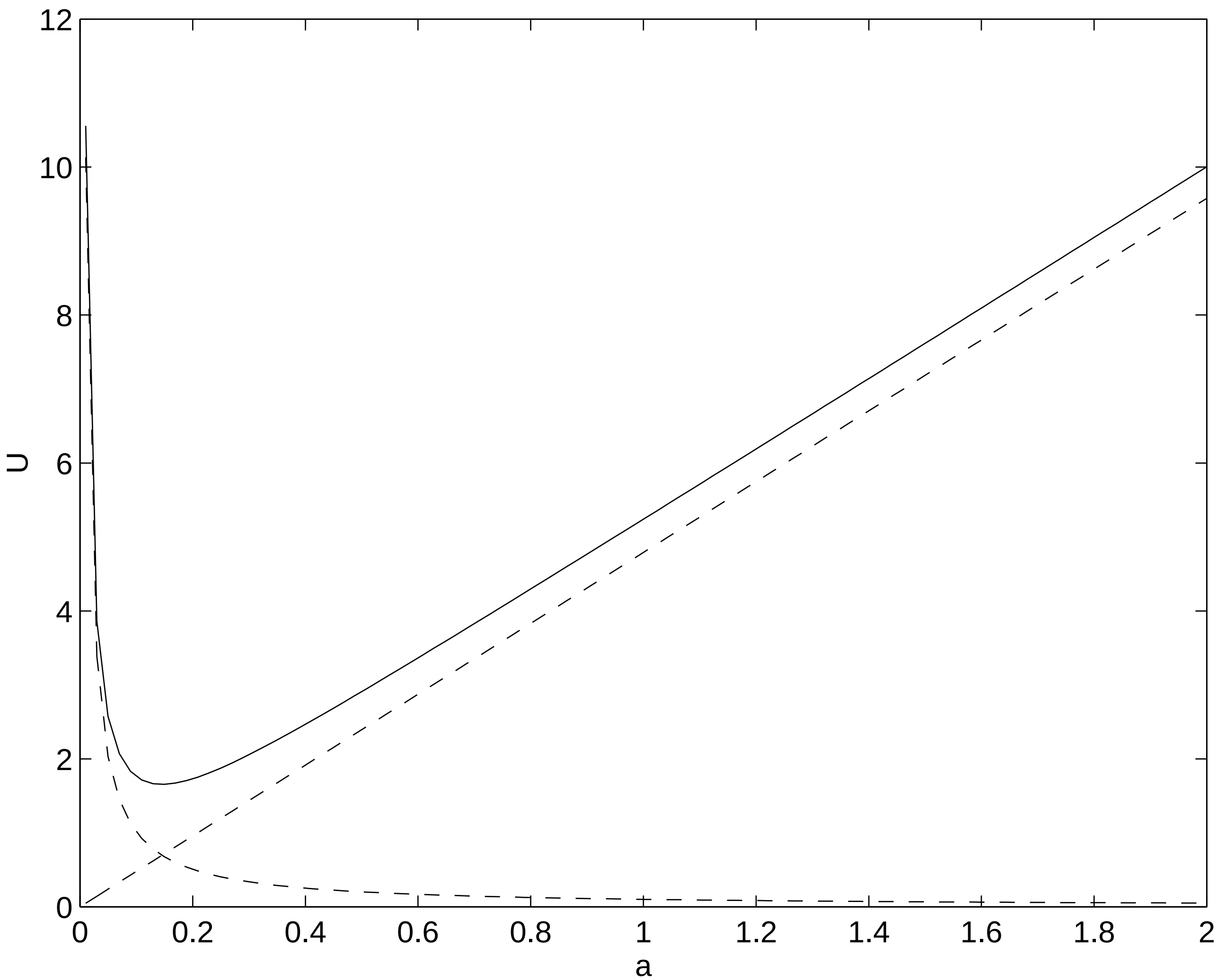
(a) (b)



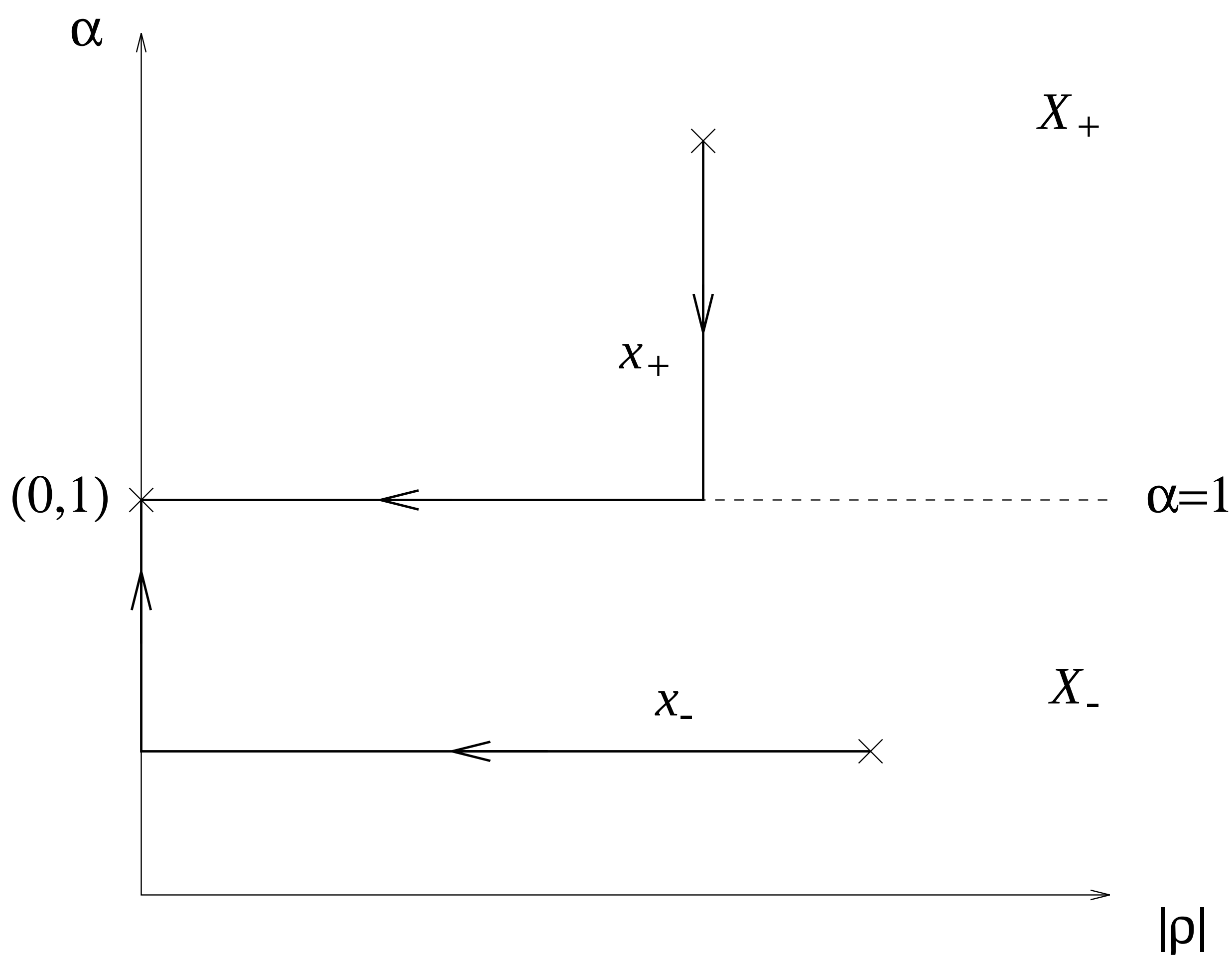

(a)
500
0
1
0.5
0
0
0.5
1
(b)
40
20
0
1
0.5
0
0
0.5
1
(c)
40
20
0
1
0.5
0
0
0.5
1
(d)
40
20
0
1
0.5
0
0
0.5
1
(e)
1000
500
0
1
0.5
0
0
0.5
1



40
30
20
10
0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1



U
a
0
2
4
6
8
10
12
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



α
X_+
x_+
(0,1)
α=1
X_-
x_-
|ρ|